Geometry of holomorphic isometric embeddings between bounded symmetric domains and applications

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Content

1. Asymptotic behaviour of local holomorphic curves
2. Applications and some related known results

These materials are based on my joint work with N. Mok (*J. Differential Geom.* 2022).

1. Asymptotic behaviour of local holomorphic curves

Theorem (C.-Mok, J. Diff. Geom. 2022)

Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain (BSD) equipped with the Bergman metric ds^2_Ω. Let $\mu : U := \mathbb{B}^1(b_0, \epsilon) \to \mathbb{C}^N$, $\epsilon > 0$, be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and $\mu(U \cap \partial\Delta) \subset \partial\Omega$, where $b_0 \in \partial\Delta$. Denote by $\sigma(z)$ the second fundamental form of $\mu(U \cap \Delta)$ in (Ω, ds^2_Ω) at $z = \mu(w)$. Then, for a general point $b \in U \cap \partial\Delta$ we have

$$\lim_{w \in U \cap \Delta,\ w \to b} \|\sigma(\mu(w))\| = 0.$$

Here, a general point b on $U \cap \partial\Delta$ means all b on the circular arc $U \cap \partial\Delta$ except for a discrete subset of $U \cap \partial\Delta$.

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Moreover, we have not obtained a precise estimate on $\|\sigma(\mu(w))\|$.
However, this theorem was obtained by N. Mok (*Pure and Appl. Math. Q.* 2014) for μ exiting at points in $\mathrm{Reg}(\partial\Omega)$ with the precise estimate of $\|\sigma(\mu(w))\|$, namely, for any neighborhood U_0 of the general point b in $\mathbb{C}$ such that $U_0 \Subset U$ and $\|\sigma(\mu(w))\|^2$ is real-analytic on U_0, there exists a real constant $C > 0$ depending on U_0 such that

$$\|\sigma(\mu(w))\| \le C\delta(w)$$

for any $w \in U_0 \cap \Delta$.

By the fact that holo. isometries extend holomorphically around a general boundary point, we have

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Let $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ *be a holomorphic isometric embedding, where* $\lambda > 0$ *is a real constant and* $\Omega \Subset \mathbb{C}^N$ *is a bounded symmetric domain. Then,* f *is asymptotically totally geodesic at a general point* $b \in \partial\Delta$*.*

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$$\varphi(w) := \|\sigma(f(w))\|^2 \leq C\delta(w)^q$$

for some constant $C > 0$, where $q = 2$ or $q = 1$, $\delta(w) := 1 - |w|$.

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for some constant $C > 0$, where $q = 2$ or $q = 1$, $\delta(w) := 1 - |w|$. Define $E(f) := \{b \in \partial\Delta : \varphi \text{ extends real-analytically around } b\}$. It is still unknown if there exists a holo. isometry $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$, $b \in E(f)$, and an open neighborhood U_b of $b \in \partial\Delta$ in $\mathbb{C}$ with φ extending real-analytically on U_b, such that

$$C'\delta(w)^2 < \varphi(w) = \|\sigma(f(w))\|^2 \le C\delta(w)$$

holds on $U_b \cap \Delta$ for some real constants $C, C' > 0$.

We will first prove the theorem when $\Omega \Subset \mathbb{C}^N$ is an irreducible bounded symmetric domain of rank r. Let $\mu : U = \mathbb{B}^1(b_0, \epsilon) \to \mathbb{C}^N$, $\epsilon > 0$, be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and $\mu(U \cap \partial\Delta) \subset \partial\Omega$, where $b_0 \in \partial\Delta$. For a general point $b \in U \cap \partial\Delta$, $\|\sigma(\mu(w))\|^2$ is real-analytic around b by Mok (2009).

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Let $\{w_k\}_{k=1}^{+\infty}$ be a sequence of points in $U \cap \Delta$ such that $w_k \to b$ as $k \to +\infty$. Let $\varphi_k \in \mathrm{Aut}(\Delta)$ be the map

$$\varphi_k(\zeta) = \frac{\zeta + w_k}{1 + \overline{w_k}\zeta} \qquad (\varphi_k(0) = w_k)$$

and $\Phi_k \in \mathrm{Aut}(\Omega)$ be such that $\Phi_k(\mu(w_k)) = \mathbf{0}$, i.e., $\Phi_k(\mu(\varphi_k(0))) = \mathbf{0}$, for $k = 1, 2, 3, \ldots$.

Holomorphic isometries via the Rescaling Argument

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Consider the sequence $\{\Phi_k \circ (\mu \circ \varphi_k)\}_{k=1}^{+\infty}$ of germs of holomorphic maps from $(\Delta; 0)$ to $(\Omega; \mathbf{0})$. All $\Phi_k \circ (\mu \circ \varphi_k)$ are defined on some small open neighborhood $U' := \mathbb{B}^1(0, \epsilon') \subset \Delta$ of 0 in Δ, where $\epsilon' > 0$.

Lemma

Let $b \in U \cap \partial\Delta$ be a general point. Choose some sequence $\{w_k\}_{k=1}^{+\infty}$ of points in $U \cap \Delta$ converging to a general point $b \in U \cap \partial\Delta$ as $k \to +\infty$. Then, after shrinking U' if necessary, there is a subsequence of $\{\widetilde{\mu}_k := \Phi_k \circ (\mu \circ \varphi_k)\}_{k=1}^{+\infty}$ which converges to some holomorphic map $\widetilde{\mu}$ on U' such that $\widetilde{\mu} : (\Delta, m_0 g_\Delta; 0) \to (\Omega, g_\Omega; \mathbf{0})$ is a germ of holomorphic isometry from some integer $m_0 \geq 1$.

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Moreover, $\widetilde{\mu}$ may be chosen such that $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 \equiv \|\sigma(\mu(b))\|^2$ is a constant function. $\widetilde{\mu}$ can be extended to a global holomorphic isometry, still denoted by $\widetilde{\mu}$.

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Moreover, $\widetilde{\mu}$ may be chosen such that $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 \equiv \|\sigma(\mu(b))\|^2$ is a constant function. $\widetilde{\mu}$ can be extended to a global holomorphic isometry, still denoted by $\widetilde{\mu}$.

Write $Z := \widetilde{\mu}(\Delta)$. At each point $w \in \Delta$, we choose a unit tangent vector $\eta(z) \in T_z(Z)$, $z := \widetilde{\mu}(w)$, and write $\xi_z := (\xi_z^1, \ldots, \xi_z^r)$ for the normal form of $\eta(z)$, where $\xi_z \in T_0(\Omega)$ is tangent to the standard maximal polydisk $\Pi \cong \Delta^r$ in Ω, and there exists $\gamma \in \mathrm{Aut}(\Omega)$ such that (a) $\gamma(z) = 0$, (b) $d\gamma(\eta(z)) = \xi_z$, and (c) $\xi_z^1 \geq \cdots \geq \xi_z^r \geq 0$ are real numbers. Then, we may further assume that $\xi_z^j = \xi^j$, $1 \leq j \leq r$, are constants independent of z.

Remark: It is clear that $\exists\ k$, $1 \leq k \leq r$, such that $\xi_z^1 \geq \cdots \geq \xi_z^k > 0$, and if $k \leq r-1$, then $\xi_z^j = 0$ for all $j \geq k+1$. Then, k is called the rank of $\eta(z)$, and k is independent of z by the lemma.
With this lemma, to obtain the asymptotic total geodesy of μ, it suffices to prove that $\|\widetilde{\sigma}\|^2 \equiv 0$, equivalently, $Z \subset \Omega$ is totally geodesic.

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Rank of a tangent vector. $\forall$ non-zero vector $v \in T_z(\Omega)$ we have the normal form of v given by $d\gamma_z(v) = (a_1, \ldots, a_r)$ that is tangent to the maximal polydisk $\cong \Delta^r$ at $\mathbf{0}$, and $a_1 \geq \cdots \geq a_r \geq 0$ are real numbers, where $\gamma \in \mathrm{Aut}(\Omega)$ with $\gamma(z) = \mathbf{0}$ (cf. Mok 1989).
One may first get $(w_1, \ldots, w_r) \in T_{\mathbf{0}}(\Pi) \cong T_{\mathbf{0}}(\Delta^r)$ for $w_j \in \mathbb{C}$, $1 \leq j \leq r$, but then we may apply the action of $(S^1)^r$ on Δ^r (as automorphisms) to get $e^{\sqrt{-1}\theta_j} w_j = a_j \geq 0$ for some $\theta_j \in [0, 2\pi)$, $1 \leq j \leq r$, and we rearrange the order of a_j's and assume $a_1 \geq \cdots \geq a_r \geq 0$. It is clear that $\exists\ k$, $1 \leq k \leq r$, such that $a_1 \geq \cdots \geq a_k > 0$ and $a_j = 0$ for all $j \geq k+1$ if $k \leq r-1$. Then, k is called the rank of v.

This lemma also yields

Proposition

Let $f_0 : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometric embedding. If $Z_0 := f_0(\Delta) \subset \Omega$ is not asymptotically totally geodesic at a general point $b \in \partial Z_0$, then there exists by rescaling a holomorphic isometric embedding $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ with the image $Z := f(\Delta)$ that is not totally geodesic in Ω, such that all holomorphic tangent spaces $T_x(Z)$, $x \in Z$, are equivalent under $\mathrm{Aut}(\Omega)$.

Therefore, our goal is to show that Z is actually totally geodesic, and thus the original holomorphic isometry f_0 must be asymptotically totally geodesic at general points.

Total geodesy of local holo. curves on Tube domains

Let Ω be an irr. BSD. In 2002, Mok (*Comp. Math.* 2002) considered $\mathcal{S} \subset \mathbb{P}T_\Omega$ defined as $\mathcal{S} := \bigcup_{x\in\Omega} \mathcal{S}_x$, where

$$\mathcal{S}_x := \{[\eta] \in \mathbb{P}T_x(\Omega) : \eta \text{ is of rank } < \mathrm{rank}(\Omega)\}.$$

Then, $\mathcal{S}_0 \subset \mathbb{P}T_0(\Omega)$ is of complex codimension 1 $\iff$ Ω is of tube type, i.e., Ω is one of the following

1. $D^{I}_{m,m}$, $m \geq 1$,
2. D^{II}_{n}, $n \geq 4$ is even,
3. D^{III}_{n}, $n \geq 3$,
4. D^{IV}_{n}, $n \geq 3$,
5. D^{VI} (27-dimensional exceptional domain pertaining to E_7).

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5. D^{VI} (27-dimensional exceptional domain pertaining to E_7).

Proposition

Let Ω be an irr. BSD of tube type and of rank r, $Z \subset \Omega$ be a local holomorphic curve with $\mathrm{Aut}(\Omega)$-equivalent holomorphic tangent spaces spanned by holomorphic tangent vectors of rank r. Then, $Z \subset \Omega$ is totally geodesic and of diagonal type, i.e. it is equivalent to the image of the map $\Delta \to \Omega$, $w \mapsto (w, \ldots, w, \mathbf{0})$.

Proof: $\pi : \mathbb{P}T_\Omega \to \Omega$, $L \to \mathbb{P}T_\Omega$ tautological line bundle. By [Mok, Comp. Math. 2002], the divisor line bundle $\mathrm{Div}(\mathcal{S})$ over $\mathbb{P}T_\Omega$ defined by the divisor $\mathcal{S} \subset \mathbb{P}T_\Omega$ is

$$\mathrm{Div}(\mathcal{S}) \cong L^{-r} \otimes \pi^* E^2,$$

where E dual to $\mathcal{O}(1)$ on the compact dual Hermitian symmetric space X_c of Ω. By the Poincaré-Lelong equation

$$\frac{1}{2\pi}\sqrt{-1}\partial\overline{\partial}\log\|s\|^2 = rc_1(L, \hat{g}_0) - 2c_1(\pi^* E, \pi^* h_0) + [\mathcal{S}],$$

where $\hat{g}_0$ and h_0 are canonical metrics, s is a holomorphic section of $L^{-r} \otimes \pi^* E^2$ such that the zero divisor of s is $\mathcal{S}$, $[\mathcal{S}]$ denotes the current of integration over $\mathcal{S}$. Now, $\|s\|$ only depends on the $\mathrm{Aut}(\Omega)$-isomorphism type of tangent vectors in $T_z(\Omega)$, $z \in \Omega$, i.e., $\|s\|$ is invariant under $\mathrm{Aut}(\Omega)$. Consider the tautological lifting $\hat{Z}$ of Z to $\mathbb{P}T_\Omega$, i.e.,

$$\hat{Z} := \{[\alpha] \in \mathbb{P}T_x(\Omega) : x \in Z,\ T_x(Z) = \mathbb{C}\alpha\}.$$

Then, $\hat{Z} \cap \mathcal{S} = \varnothing$.

Moreover, since Z has $\mathrm{Aut}(\Omega)$-equivalent holomorphic tangent spaces, $\|s\| \equiv \text{Constant} > 0$ on $\hat{Z}$, and thus

$$0 \equiv rc_1(L, \hat{g}_0)|_{\hat{Z}} - 2c_1(\pi^* E, \pi^* h_0)|_{\hat{Z}}$$

so that

$$0 \equiv rc_1(T_Z, g_\Omega|_Z) - 2c_1(E, h_0)|_Z,$$

which is equivalent to the Gaussian curvature $K(x) = -\frac{2}{r}$, and thus the second fundamental form σ of Z is 0, $\sigma \equiv 0$. □

Inserting a totally geodesic complex submanifold $\Omega' \supset Z$

Proposition

Let Ω be an irr. BSD, $Z \subset \Omega$ be a local holomorphic curve with $\mathrm{Aut}(\Omega)$-equivalent tangent spaces $T_z(Z) = \mathbb{C}\eta_z$. Suppose $\mathrm{rank}(\eta_z) =: k < r := \mathrm{rank}(\Omega)$. Then, there exists a holomorphic vector bundle $W \subset T_\Omega|_Z$ such that

1. *defining the second fundamental form $\tau : T_Z \otimes W \to T_\Omega|_Z / W$ of W in $T_\Omega|_Z$ by*

$$\tau_x(\eta \otimes \gamma) := (\nabla_\eta \gamma)(x) \mod W_x$$

   for $x \in Z$, $\eta \in T_x(Z)$ and $\gamma \in W_x$, τ is holomorphic, i.e., $\nabla_{\overline{\beta}}(\nabla_\eta \gamma)(x) \in W_x$ for any $(1,0)$-tangent vector β of Z at x.

2. *We have $\tau|_{T_Z \otimes T_Z} \equiv 0$, and indeed $\tau \equiv 0$, i.e., W is parallel on Z.*
3. *there exists a totally geodesic complex submanifold $\Omega' \subset \Omega$ such that $Z \subset \Omega'$ and $T_z(\Omega') = W_z$ for all $z \in Z$.*
4. *Ω' is an irreducible BSD and $\mathrm{rank}(\Omega') = k < \mathrm{rank}(\Omega)$.*

Construction of W $\to$ Obtain $\Omega' \supset Z$ via the method of holo. foliations.

Remark: After this proposition, we still need to consider the case where $\mathrm{rank}(\eta_z) = r = \mathrm{rank}(\Omega)$, and Ω is not of tube type. If Ω is of tube type, then we may apply the propositions on pages 9 & 12. Thus, we need to have a similar result that forces $Z \subset \Omega'$ for some totally geodesic complex submanifold $\Omega' \subset \Omega$ such that $Z \subset \Omega'$, Ω' is of tube type and $\mathrm{rank}(\Omega') = r$.

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Construction of the vector bundle W. For any $z \in \Omega$ define the Hermitian bilinear form on $T_z(\Omega) \otimes \overline{T_z(\Omega)}$ by

$$Q_z(\alpha \otimes \overline{\beta}, \gamma \otimes \overline{\delta}) := R_{\alpha\overline{\gamma}\delta\overline{\beta}}(\Omega, g_\Omega).$$

In the following we simply write the curvature as $R_{\alpha\overline{\gamma}\delta\overline{\beta}} = R(\alpha, \overline{\gamma}, \delta, \overline{\beta})$. Note that $Q_z(\alpha \otimes \overline{\beta}, \cdot) = R_{\alpha\overline{*}*\overline{\beta}}$. For any non-zero vector $\xi \in T_z(\Omega)$, we define the null space

$$\mathcal{N}_\xi := \{v \in T_z(\Omega) : Q_z(\xi \otimes \overline{v}, \cdot) \equiv 0\}.$$

For any $x \in Z$, we define

$$W_x := \{v \in T_x(\Omega) : Q_x(v \otimes \overline{\zeta}, \cdot) \equiv 0 \quad \forall\, \zeta \in \mathcal{N}_\eta\},$$

where $\eta = \eta_x \in T_x(Z)$ is a non-zero vector spanning $T_x(Z)$. It is clear that $Q_x(\eta \otimes \overline{\zeta}, \cdot) \equiv 0$ for all $\zeta \in \mathcal{N}_\eta$ by definition, hence $T_x(Z) \subset W_x$.

Example. When $\Omega = D^{I}_{p,q}$, $2 \le p \le q$, Ω is of rank p, we may write

$$\eta = \operatorname{diag}_{p,q}(\eta_1, \ldots, \eta_k, \mathbf{0})$$

in the normal form with $\eta_1 \ge \cdots \ge \eta_k > 0$, where $1 \le k < p$. Then,

$$\mathcal{N}_\eta = \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Z' \end{bmatrix} \in M(p, q; \mathbb{C}) : Z' \in M(p-k, q-k; \mathbb{C}) \right\}$$

Then, W_x is isomorphic to

$$\bigcap_{\zeta \in \mathcal{N}_\eta} \mathcal{N}_\zeta = \left\{ \begin{bmatrix} Z'' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in M(p, q; \mathbb{C}) : Z'' \in M(k, k; \mathbb{C}) \right\} \cong T_{\mathbf{0}}(D^{I}_{k,k}).$$

If $k = p$, i.e., $\eta = \operatorname{diag}_{p,q}(\eta_1, \ldots, \eta_p)$, then $\mathcal{N}_\eta = \mathbf{0}$ and

$$W_x \cong \bigcap_{\zeta \in \mathcal{N}_\eta} \mathcal{N}_\zeta = \mathcal{N}_0 = T_{\mathbf{0}}(\Omega),$$

so that $W_x = T_x(\Omega)$, which actually holds for any irr. BSD Ω of rank ≥ 2 whenever $\operatorname{rank}(\eta) = \operatorname{rank}(\Omega)$. If Ω is not of tube type, we couldn't apply the proposition on page 9 when η is of rank $r = \operatorname{rank}(\Omega)$.

Inserting a totally geodesic complex submanifold $\Omega' \supset Z$

Due to the issue mentioned at the end of the previous example, we will need the following proposition to deal with the case where η_x is of max. rank r. (The idea is similar to the previous proposition.)

Proposition

Let Ω be an irr. BSD, and $Z \subset \Omega$ be a local holomorphic curve with $\mathrm{Aut}(\Omega)$-equivalent tangent spaces $T_x(Z) = \mathbb{C}\eta_x$, $x \in Z$. Then, there exists a holomorphic vector subbundle $V \subset T_\Omega|_Z$ such that

1. *defining the second fundamental form $\tau : T_Z \otimes V \to T_\Omega|_Z/V$, τ is holomorphic.*
2. *$\tau \equiv 0$, i.e., V is parallel on Z.*
3. *there exists a totally geodesic complex submanifold $\Omega' \subset \Omega$ such that $Z \subset \Omega'$, $T_x(\Omega') = V_x$ for all $x \in Z$,*
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4. *Ω' is an irreducible BSD of tube type.*

For any $x \in Z$, $V = \bigcup_{x\in Z} V_x$ is defined by

$$V_x = [[T_x(Z), \overline{T_x(\Omega)}], T_x(Z)] \subset T_x(\Omega).$$

We use the Lie algebraic properties of $T_z(\Omega)$, $z \in \Omega \cong G_0/K$.

Comparison between the holo. vector bundles V and W

Recall $T_x(Z) = \mathbb{C}\eta_x$ and η_x is of rank $k \leq r := \mathrm{rank}(\Omega) \geq 2$, $x \in Z$. If $k < r$, then we have $V_x = W_x$ for $x \in Z$, so that we can just identify $V = W$, and we can find an irreducible BSD Ω' of tube type and of rank k containing Z by the method before.

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However, if $k = r$, then $V_x \subsetneq W_x = T_x(\Omega)$ for $x \in Z$. Actually, in this case $V_x = T_x(\Omega')$ for some totally geodesic complex submanifold $\Omega' \subset \Omega$ of the same rank as Ω, and Ω' is an irreducible BSD of tube type. Thus, the key point is to deal with the case where $k = r$ and make use of V. We need some extra computations regarding those assertions on V. But most arguments in our consideration of W also work here.

From the previous two propositions, we can always find a totally geodesic complex submanifold $\Omega' \subset \Omega$ such that

1. $Z \subset \Omega'$,
2. Ω' is an irreducible BSD of tube type and rank k,
3. $T_x(Z) = \mathbb{C}\eta_x$ with $\eta_x \in T_x(\Omega')$ being a rank-k vector.

This allows us to prove that $Z \subset \Omega'$ is totally geodesic by using the proposition on page 9, and thus we prove the asymptotic total geodesy of the local holomorphic curve $\mu(U \cap \Delta)$ exiting the irreducible BSD Ω.

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When Ω is reducible, we can apply similar constructions of a holomorphic curve Z with $\mathrm{Aut}(\Omega)$-equivalent holomorphic tangent spaces, and the (holomorphic) vector bundles W and V over Z, etc.

One of the consequences of our results is the following.

Theorem (C.-Mok, J. Diff. Geom. 2022)

Let D and Ω be bounded symmetric domains, $\Phi : \mathrm{Aut}_0(D) \to \mathrm{Aut}_0(\Omega)$ be a group homomorphism, and $F : D \to \Omega$ be a Φ-equivariant holomorphic map. Then, $F(D) \subset \Omega$ is a totally geodesic complex submanifold with respect to the Bergman metric ds^2_Ω.

Remark: This theorem is due to L. Clozel (2007) in the cases of classical domains, and is stated in a survey article of N. Mok (2011).

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Remark: This theorem is due to L. Clozel (2007) in the cases of classical domains, and is stated in a survey article of N. Mok (2011).

Idea of the proof: As in the study of holo. isometries, the key point is to deal with the case where $D \cong \Delta$ is the unit disk by using N. Mok's proof of the Hermitian metric rigidity (in general we restrict F to any minimal disk of D). Now, we consider $D \cong \Delta$. Write σ for the $(1,0)$-part of second fundamental form of $(F(D), ds^2_\Omega|_{F(D)}) \subset (\Omega, ds^2_\Omega)$. By the Φ-equivariance of F, the norm $\|\sigma\|$ is constant. On the other hand, we have $\|\sigma(\mu(w))\| \to 0$ as $w \to b$, $w \in U \cap \Delta$, where μ is the local holomorphic curve defined in the theorem before. This forces $\|\sigma\| \equiv 0$, and thus $F(D) \subset \Omega$ is totally geodesic.

The hyperbolic Ax-Lindemann-Weierstrass conjecture

Another application is related to the following hyperbolic Ax-Lindemann -Weierstrass conjecture (which is related to the André-Oort conjecture).

Conjecture (The hyperbolic Ax-Lindemann-Weierstrass conjecture)

Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain and $X_\Gamma := \Omega/\Gamma$ with the universal covering map $\pi : \Omega \to X_\Gamma$, where $\Gamma \subset \mathrm{Aut}_0(\Omega)$ is a torsion-free lattice. If $Z \subset \Omega$ is an algebraic subset, then the Zariski closure $Y := \overline{\pi(Z)}^{\mathrm{Zar}}$ of $\pi(Z)$ in X_Γ is a totally geodesic subset.

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The above conjecture has been solved by (1) Klingler-Ullmo-Yafaev (2016) if X_Γ is a pure Shimura variety, i.e., Γ is arithmetic, and by (2) N. Mok (*Compos. Math.* 2019) if $\Omega \cong \mathbb{B}^N$ (Γ is not necessarily arithmetic). The general case is still open.

On the other hand, Ziyang Gao (2017) also extended this result, which is called the Ax-Lindemann principle, to any mixed Shimura variety [See a survey article of Klingler-Ullmo-Yafaev (2018)].

By the Margulis Arithmeticity Theorem, Γ is arithmetic if $\mathrm{rank}(\Omega) \geq 2$ and Ω/Γ is an irreducible quotient (i.e., Γ is irreducible).

In particular, the hyperbolic Ax-Lindemann-Weierstrass conjecture is solved if Γ is irreducible, which holds true if Ω is irreducible. Note that in general there could be non-arithmetic quotients if Ω has some irreducible factor $\cong \mathbb{B}^n$ (e.g. for $n = 2$ or 3).

Main Theorem

Theorem (C.-Mok, J. Diff. Geom. 2022; Main Theorem)

Let $\Omega \Subset \mathbb{C}^N$ be a BSD, and $Z \subset \Omega$ be an irr. algebraic subset. Suppose $\exists$ a torsion-free discrete subgroup $\check{\Gamma} \subset \mathrm{Aut}(\Omega)$ such that $\check{\Gamma}$ stabilizes Z and $\check{Y} := Z/\check{\Gamma}$ is compact (without boundary). Then, $Z \subset \Omega$ is totally geodesic.

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As a consequence, we have the following theorem that generalizes the cocompact case of Ullmo-Yafaev (2011) which characterizes totally geodesic subsets of Hermitian locally symmetric spaces of finite volume as the unique bi-algebraic subvarieties (thus yielding a reduction of the hyperbolic Ax-Lindemann-Weierstrass conjecture).

Theorem (C.-Mok, J. Diff. Geom. 2022)

Let $\Omega \Subset \mathbb{C}^N$ be a BSD, and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a not necessarily arithmetic torsion-free cocompact lattice. Write $X_\Gamma := \Omega/\Gamma$, $\pi : \Omega \to X_\Gamma$ for the uniformization map. Let $Y \subset X_\Gamma$ be an irr. subvariety, and $Z \subset \Omega$ be an irr. component of $\pi^{-1}(Y)$. Suppose $Z \subset \Omega$ is an algebraic subset. Then, $Z \subset \Omega$ is a totally geodesic complex submanifold.

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This theorem could be generalized to the case where Γ is a lattice (not necessarily cocompact) provided that Problem 1 is solved.

Idea of the proof of the main theorem

We first assume that $\check{Y} = Z/\check{\Gamma}$ is quasi-projective instead of compact. Let H_0 be the identity component of $\mathrm{Stab}(Z) := \{g \in G_0 : g(Z) = Z\}$, where $G_0 := \mathrm{Aut}_0(\Omega)$. We show that $H_0 \subset G_0$ is real algebraic group of positive dimension. Actually, since $\mathrm{Stab}(Z)$ is a real algebraic group and $\check{\Gamma} \subset \mathrm{Stab}(Z)$, we only need to show that $\check{\Gamma}$ is an infinite group by the maximum principle.

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Moreover, $\hat{\Gamma} := H_0 \cap \check{\Gamma} \subset \check{\Gamma}$ is a subgroup of finite index. In particular, we have a finite unramified covering map $Z/\hat{\Gamma} \to Z/\check{\Gamma}$. Hence, if $Z/\check{\Gamma}$ is compact, then so is $Z/\hat{\Gamma}$. In the proof, we will consider the compact complex manifold $Z/\hat{\Gamma}$ instead of $Z/\check{\Gamma}$.

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If $Z/\check{\Gamma}$ is quasi-projective, then so is $Z/\hat{\Gamma}$ by Riemann's existence theorem and the fact that $\check{\Gamma}$ acts on Z without fixed points (cf. Remark 1.3 on p. 2082 of [R. Friedman & R. Laza, *Duke Math. J.* 2013]).
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We have the complexification $H \subset G := G_0^{\mathbb{C}}$ of H_0, and H is a complex algebraic group. Here, $X_c = G/P$ is the compact dual Hermitian symmetric space of Ω and we can identify $\Omega \subset X_c$ as an open subset via the Borel embedding, where $P \subset G$ is some parabolic subgroup.

We have the following proposition by using the maximum principle.

Proposition

For $x \in Z$, $Z \subset Hx \cap \Omega$ is an irreducible component. (*Recall* $\Omega \Subset \mathbb{C}^N$.)

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Outline of the proof: By definition, $Z \subset \hat{Z} \cap \Omega$ is an irreducible component for some irreducible projective subvariety $\hat{Z} \subset X_c$. If $Hx \cap Z \subsetneq Z$, then we can find a Zariski closed subset $E \subsetneq Z$ (i.e., E is a finite union of irreducible components of $\widehat{E} \cap \Omega$ for some projective subvariety $\widehat{E} \subset X_c$) such that $Hx \cap Z \subset E$, and a polynomial $p(z)$ in $z \in \mathbb{C}^N$ such that $p|_{\widehat{E} \cap \mathbb{C}^N} \equiv 0$ and $p|_{\widehat{Z} \cap \mathbb{C}^N} \not\equiv 0$.

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The previous proposition implies that $Z \subset \Omega$ is smooth by the smoothness of Hx and that $\Omega \subset X_c$ is an open subset. We also have the (real) orbit $H_0 x \subset Z \subset Hx$. Since $\check{Y} = Z/\check{\Gamma}$ (equipped with the Kähler metric $g_{\check{Y}}$ induced from $ds^2_\Omega|_Z$) is a compact Kähler manifold with ample canonical line bundle $K_{\check{Y}}$, we may make use of a consequence of Nadel's semisimplicity theorem [Nadel, *Ann. of Math.* 1990] to obtain that H_0 is a semisimple Lie group of the noncompact type (i.e., without compact factors).

Theorem (Nadel's semisimplicity theorem , Ann. of Math. 132 (1990))

Let X be a compact Kähler manifold with ample canonical line bundle K_X, and denote by $\pi : \widetilde{X} \to X$ the uniformization map. Then, $\mathrm{Aut}_0(\widetilde{X})$ is a semisimple Lie group of the noncompact type.

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Now, H_0 is a semisimple Lie group of the noncompact type. To prove the main theorem, we will show that $Z = H_0x$ is Riemannian symmetric of the semisimple and noncompact type by showing that $\dim_{\mathbb{R}}(H_0x) = \dim_{\mathbb{R}}(Z)$ and $(H_0)_x \subset H_0$ is the maximal compact subgroup. In particular, we see that $Z \subset \Omega$ is a totally geodesic complex submanifold, and the main theorem will follow.

Outline of the proof: We have $H_0x \cong H_0/(H_0)_x$, where $(H_0)_x := \{h \in H_0 : h(x) = x\}$. Note that $(H_0)_x \subset \{g \in G_0 : g(x) = x\} =: K_x$ and $K_x \subset G_0$ is known to be a maximal compact subgroup. Now, there is a maximal compact subgroup $L \subset H_0$ such that $(H_0)_x \subset L$, and $H_0/L \cong \mathbb{R}^n$ is a diffeomorphism for some n.

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$$g : S_{\hat{\Gamma}} := \hat{\Gamma} \backslash H_0/L \hookrightarrow \hat{\Gamma} \backslash \Omega \cong \Omega/\hat{\Gamma} =: X_{\hat{\Gamma}}$$

and the inclusion map $\iota : \hat{Y} := Z/\hat{\Gamma} \hookrightarrow X_{\hat{\Gamma}}$. We have the finite covering $\hat{Y} \to \check{Y}$ and $\check{Y}$ is compact (without boundary), thus $\hat{Y}$ is compact.

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$$(g_*)^{-1} \circ \iota_* : \pi_1(\hat{Y}) \to \pi_1(S_{\hat{\Gamma}}).$$

Then, there is a continuous map $f : \hat{Y} \to S_{\hat{\Gamma}}$ such that

$$f_* = (g_*)^{-1} \circ \iota_*$$

by Whitehead's theorem. Letting $g \circ f : \hat{Y} \to X_{\hat{\Gamma}}$, we have

$$(g \circ f)_* = g_* \circ f_* = \iota_*.$$

By Whitehead's theorem and Whitney's approximation theorem, we may choose f to be smooth and we have the homotopic smooth maps

$$g \circ f : \hat{Y} \to X_{\hat{\Gamma}}, \qquad \iota : \hat{Y} \hookrightarrow X_{\hat{\Gamma}}.$$

These two smooth maps induce the same pullback maps on the de Rham cohomology groups

$$(g \circ f)^* = \iota^* : H^p_{\mathrm{dR}}(X_{\hat{\Gamma}}) \to H^p_{\mathrm{dR}}(\hat{Y})$$

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for all p. Write $\hat{\omega}$ for the Kähler form of $X_{\hat{\Gamma}} = \Omega/\hat{\Gamma}$ with the Kähler metric $g_{X_{\hat{\Gamma}}}$ induced from ds^2_{Ω}. Write $s := \dim_{\mathbb{C}}(\hat{Y}) = \dim_{\mathbb{C}}(Z)$. Then,

$$\iota^* \frac{\hat{\omega}^s}{s!} = (g \circ f)^* \frac{\hat{\omega}^s}{s!} + d\eta_0$$

and $\iota^* \frac{\hat{\omega}^s}{s!}$ is the volume form of the compact Kähler manifold $(\hat{Y}, g_{X_{\hat{\Gamma}}}|_{\hat{Y}})$.

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These two smooth maps induce the same pullback maps on the de Rham cohomology groups

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for all p. Write $\hat{\omega}$ for the Kähler form of $X_{\hat{\Gamma}} = \Omega/\hat{\Gamma}$ with the Kähler metric $g_{X_{\hat{\Gamma}}}$ induced from ds^2_{Ω}. Write $s := \dim_{\mathbb{C}}(\hat{Y}) = \dim_{\mathbb{C}}(Z)$. Then,

$$\iota^* \frac{\hat{\omega}^s}{s!} = (g \circ f)^* \frac{\hat{\omega}^s}{s!} + d\eta_0$$

and $\iota^* \frac{\hat{\omega}^s}{s!}$ is the volume form of the compact Kähler manifold $(\hat{Y}, g_{X_{\hat{\Gamma}}}|_{\hat{Y}})$. If $\dim_{\mathbb{R}}(S_{\hat{\Gamma}}) < 2s$, then $g^* \hat{\omega}^s = 0$ so that

$$\iota^* \frac{\hat{\omega}^s}{s!} = d\eta_0$$

on $\hat{Y}$, and we would have $\mathrm{Vol}(\hat{Y}) = 0$ by Stokes' Theorem, a plain contradiction. Therefore, $\dim_{\mathbb{R}}(S_{\hat{\Gamma}}) \geq 2s$.

We have

$$\dim_{\mathbb{R}}(H_0x) \geq \dim_{\mathbb{R}}(H_0/L) = \dim_{\mathbb{R}}(S_{\hat{\Gamma}}) \geq 2s = \dim_{\mathbb{R}}(Z)$$

Thus, $\dim_{\mathbb{R}}(H_0x) = \dim_{\mathbb{R}}(Z)$ so that $Z = H_0x \cong H_0/L$ is Riemannian symmetric of the semisimple and noncompact type. Since Z is a complex manifold, Z is indeed a Hermitian symmetric space of noncompact type. By the theorem on page 18 about equivariant holomorphic maps, we obtain the total geodesy of Z in Ω. □

Remark: In this proof, one may consider the case where $\check{Y} = Z/\check{\Gamma}$ is only assumed quasi-projective so that $\hat{Y} = Z/\hat{\Gamma}$ is also quasi-projective. Then, we still obtain $\iota^* \frac{\hat{\omega}^s}{s!} = d\eta_0$. However, we could not apply Stokes' Theorem in the in order to do the dimension estimates as in the case where $\check{Y}$ (resp. $\hat{Y}$) is compact.

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Remark: In this proof, one may consider the case where $\check{Y} = Z/\check{\Gamma}$ is only assumed quasi-projective so that $\hat{Y} = Z/\hat{\Gamma}$ is also quasi-projective. Then, we still obtain $\iota^* \frac{\hat{\omega}^s}{s!} = d\eta_0$. However, we could not apply Stokes' Theorem in the in order to do the dimension estimates as in the case where $\check{Y}$ (resp. $\hat{Y}$) is compact. Another issue is that if $\check{Y} = Z/\check{\Gamma}$ is quasi-projective and noncompact, then we could not apply Nadel's semisimplicity theorem to show that H_0 is semisimple. However, there could be other ways to prove the semisimplicity of H_0.

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Thank you!