Geometry of holomorphic isometric embeddings between bounded symmetric domains and applications

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Content

1. Background on bounded symmetric domains and holomorphic isometries
2. Results for holomorphic isometries between bounded symmetric domains

1. Background

Definition (Bounded Symmetric Domains)

A bounded domain $D \Subset \mathbb{C}^N$ $(N < +\infty)$ is a **bounded symmetric domain** (BSD) $\iff$ $\forall\, x \in D$, $\exists$ a biholomorphism $\sigma_x : D \to D$ such that $\sigma_x^2 = \mathrm{Id}_D$ and x is an isolated fixed point of σ_x.

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(1) Hermitian symmetric spaces of the noncompact type are precisely bounded symmetric domains via the **Harish-Chandra embeddings**. **(2)** Write $\mathrm{rank}(D)$ for the rank of a BSD D as a Riemannian (globally) symmetric space, namely, $\mathrm{rank}(D)$ is the maximal dimension of a flat (i.e., curvature tensor $\equiv 0$) totally geodesic submanifold of D. We refer to the book "*Differential geometry, Lie groups, and symmetric spaces*" written by S. Helgason.

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Example: The complex unit n-ball in $\mathbb{C}^n$ defined by

$$\mathbb{B}^n := \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\| < 1 \right\}$$

is a BSD of rank 1, where $\|z\|^2 := \sum_{j=1}^n |z_j|^2$. Denote by $\Delta := \mathbb{B}^1$ the (open) unit disk on the complex plane $\mathbb{C}$.

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Let $D \Subset \mathbb{C}^m$ be a BSD. Then, we have the following basic facts.

1. D is of rank 1 $\iff$ D is biholomorphic to $\mathbb{B}^m$.
2. The boundary ∂D is smooth $\iff$ $D \cong \mathbb{B}^m$ is of rank 1.

Classical type: For $p, q \geq 1$,

$$D^{\mathrm{I}}_{p,q} := \left\{ Z \in M(p,q;\mathbb{C}) : I_q - \overline{Z}^t Z > 0 \right\}, \quad \mathrm{rank}(D^{\mathrm{I}}_{p,q}) = \min\{p,q\}.$$

$$D^{\mathrm{II}}_{m} := \left\{ Z \in D^{\mathrm{I}}_{m,m} : Z = -Z^t \right\},\ m \geq 2, \quad \mathrm{rank}(D^{\mathrm{II}}_{m}) = \left\lfloor \frac{m}{2} \right\rfloor,$$

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$$D^{\mathrm{IV}}_{n} := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 2,\ \sum_{j=1}^n |z_j|^2 < 1 + \left| \frac{1}{2} \sum_{j=1}^n z_j^2 \right|^2 \right\}$$

for $n \geq 3$, and $\mathrm{rank}(D^{\mathrm{IV}}_{n}) = 2$. We have $D^{\mathrm{I}}_{1,n} \cong D^{\mathrm{I}}_{n,1} \cong \mathbb{B}^n$.

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Exceptional type:
D^{V} of type E_6 with $\dim_{\mathbb{C}}(D^{\mathrm{V}}) = 16$ and $\mathrm{rank}(D^{\mathrm{V}}) = 2$;
D^{VI} of type E_7 with $\dim_{\mathbb{C}}(D^{\mathrm{VI}}) = 27$ and $\mathrm{rank}(D^{\mathrm{VI}}) = 3$.
One may express these irr. BSDs of exceptional type in terms of the Jordan algebra (see F. Viviani 2014).

Bergman metrics on bounded domains

Let $\Omega \Subset \mathbb{C}^N$ be a bounded domain, and $H^2(\Omega)$ be the space of square-integrable (i.e., L^2) holomorphic functions with respect to the Lebesgue measure $d\lambda$. Then, $H^2(\Omega)$ is a Hilbert space. Letting $\{h_j\}_{j=1}^{+\infty}$ be an orthonormal basis of $H^2(\Omega)$, the Bergman kernel $K_\Omega(z,w)$ of Ω is defined by

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Proposition (See Ch. 4 in the book written by N. Mok in 1989)

The above infinite sum converges uniformly on compact sets to a real-analytic function in (z, w) such that $K_\Omega(z, w)$ is holomorphic in z, anti-holomorphic in w, independent of the choice of the basis $\{h_j\}_{j=1}^{+\infty}$, and possesses a reproducing property for L^2 holomorphic function f given by $f(z) = \int_\Omega K_\Omega(z, w)f(w)d\lambda(w)$.

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This defines a canonical Kähler metric ds_Ω^2 on Ω, called the **Bergman metric**, and its Kähler form is

$$\omega_{ds_\Omega^2} = \sqrt{-1}\partial\overline{\partial}\log K_\Omega(z,z).$$

Canonical Kähler metrics on irr. BSDs

Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain. Then, it is well known that

$$K_\Omega(z, w) = \frac{1}{\mathrm{Vol}_{\mathrm{Euc}}(\Omega)} \cdot \frac{1}{h_\Omega(z, w)^{p(\Omega)+2}}$$

for some polynomial $h_\Omega(z, w)$ in $(z, \overline{w})$, where $z, w \in \mathbb{C}^N$, $p(\Omega)$ is a nonnegative integer depending on Ω, and $\mathrm{Vol}_{\mathrm{Euc}}(\Omega)$ is the Euclidean volume of Ω with respect to the Lebesgue measure.

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Example: We have $\omega_{ds^2_{D^{\mathrm{I}}_{p,q}}} = -(p+q)\sqrt{-1}\partial\overline{\partial}\log\det\left(I_q - \overline{Z}^t Z\right)$, $p(D^{\mathrm{I}}_{r,s}) = r+s-2$ and $\omega_{g_{D^{\mathrm{I}}_{r,s}}} = -\sqrt{-1}\partial\overline{\partial}\log\det\left(I_s - \overline{Z}^t Z\right)$.

Let $D \Subset \mathbb{C}^n$ be a bounded symmetric domain (BSD) (containing the origin $\mathbf{0}$). Then, $D \cong G_0/K$ is a homogeneous space, where $G_0 := \mathrm{Aut}_0(D)$ and $K := \{g \in G_0 : g(\mathbf{0}) = \mathbf{0}\}$ is the isotropy subgroup of G_0 at $\mathbf{0}$.

Characteristic symmetric subspaces

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On the other hand, a non-zero vector $\alpha \in T_x^{1,0}(D)$, $x \in D$, is called a **characteristic vector** if it realizes the minimum of the holomorphic sectional curvature of (D, ds_D^2). It is well known that $\exists$ a totally geodesic unit disk $\Delta \cong \Sigma \subset D$ passing through $x \in D$ such that $T_x^{1,0}(\Sigma) = \mathbb{C}\alpha$. Such a subspace Σ is called a **minimal disk** of D (Mok-Tsai 1992).

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Example: The CSSs of $D^{\mathrm{I}}_{p,q}$, $p \geq q \geq 2$, are biholomorphic to images of $\iota_k : D^{\mathrm{I}}_{p-k,q-k} \hookrightarrow D^{\mathrm{I}}_{p,q}$, $\iota_k(Z) := \begin{bmatrix} Z & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $Z \in D^{\mathrm{I}}_{p-k,q-k}$, $1 \leq k \leq q-1$. Here, $\mathrm{rank}(D^{\mathrm{I}}_{p,q}) = q$.

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For any BSD $D \Subset \mathbb{C}^n$, we have the **Polydisk Theorem** (cf. Wolf 1972 and N. Mok 1989) which says that there exists a totally geodesic complex submanifold $\Pi \subset D$ such that $\Pi \cong \Delta^r$, called the *maximal polydisk*, and

$$D = \bigcup_{k \in K} k \cdot \Pi.$$

When D is irreducible, $(\Pi, ds_D^2|_\Pi)$ is holomorphically isometric to $(\Delta^r, \lambda ds_{\Delta^r}^2)$ for some rational number $\lambda > 0$ such that the max. holomorphic sectional curvature of $(\Delta^r, \lambda ds_{\Delta^r}^2)$ is the same as that of (D, ds_D^2).

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$$h_k(\zeta) := (w_1, \ldots, w_{k-1}, \zeta, w_{k+1}, \ldots, w_r).$$

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We also have the **Hermann Convexity Theorem** (cf. Wolf 1972 and N. Mok 1989) which says that $D \Subset \mathbb{C}^n$ is a bounded convex domain.

Definition (Proper holomorphic maps between bounded domains)

Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded domains. Then, a holomorphic map $F : D \to \Omega$ is **proper** $\iff$ $\forall$ sequence $\{z_j\}_{j=1}^{+\infty} \subset D$ with $\lim_{j\to+\infty} \mathrm{dist}_{\mathrm{Euc}}(z_j, \partial D) = 0$, we have $\lim_{j\to+\infty} \mathrm{dist}_{\mathrm{Euc}}(F(z_j), \partial\Omega) = 0$.

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Let (X, h_X) and (Y, h_Y) be Kähler manifolds with Kähler forms ω_{h_X} and ω_{h_Y} respectively. A holomorphic map $F : (X, h_X) \to (Y, h_Y)$ is a **holomorphic isometry** if and only if $F^*\omega_{h_Y} = \omega_{h_X}$.

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In this definition, if $F : (X; x_0) \to (Y; y_0)$ is only a germ of holomorphic map, $y_0 = F(x_0)$, we call F a germ of holomorphic isometry.

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The study of holomorphic isometries between Kähler manifolds was started by E. Calabi (*Ann. of Math.* 1953) under the supervision of S. Bochner. E. Calabi also introduced the *diastasis* (or *diastatic function*) on any Kähler manifold with a real-analytic Kähler metric g, and proved that the *diastasis* is uniquely determined by g, so that a holo. isometry between such manifolds preserves the diastatic functions.

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Let (X, h_X) and (Y, h_Y) be Kähler manifolds with Kähler forms ω_{h_X} and ω_{h_Y} respectively. A holomorphic map $F : (X, h_X) \to (Y, h_Y)$ is a **holomorphic isometry** if and only if $F^*\omega_{h_Y} = \omega_{h_X}$.

In this definition, if $F : (X; x_0) \to (Y; y_0)$ is only a germ of holomorphic map, $y_0 = F(x_0)$, we call F a germ of holomorphic isometry.
The study of holomorphic isometries between Kähler manifolds was started by E. Calabi (*Ann. of Math.* 1953) under the supervision of S. Bochner. E. Calabi also introduced the *diastasis* (or *diastatic function*) on any Kähler manifold with a real-analytic Kähler metric g, and proved that the *diastasis* is uniquely determined by g, so that a holo. isometry between such manifolds preserves the diastatic functions.

Example: For a bounded domain $U \Subset \mathbb{C}^n$ with the Bergman metric ds_U^2, the *diastasis* is given by

$$\delta_U(x, y) = \log \frac{K_U(x,x)K_U(y,y)}{K_U(x,y)K_U(y,x)},$$

where K_U is the Bergman kernel of U.

Holomorphic isometries

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For any germ $f : (U, \lambda ds_U^2; 0) \to (V, ds_V^2; 0)$, $\lambda > 0$, of holo. isometries between certain bounded domains $U \Subset \mathbb{C}^n$ and $V \Subset \mathbb{C}^N$, N. Mok (2012) has shown that f satisfies the so-called *polarized functional equation*

$$K_V(f(z), f(w)) = A \cdot K_U(z, w)^\lambda$$

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A germ of holo. isometry between BSDs extends to a global holo. isometric embedding F by N. Mok (2012) [the global extension also follows from E. Calabi (1953)]. Indeed, N. Mok (2012) has proved that the holo. isometry F is algebraic in the sense that $\mathrm{Graph}(F)$ extends as an affine-algebraic subvariety of $\mathbb{C}^n \times \mathbb{C}^N$.

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Idea: In the proof of Mok's extension theorem (for germs of holo. isometries f between BSDs D and Ω, $f(0) = 0$), the key point is to study the system of functional equations

$$K_\Omega(\xi, f(z)) = A \cdot K_D(\zeta, z)^\lambda, \quad (\zeta, \xi) \in D \times \Omega, \tag{1}$$

for all $z \in D_\varepsilon$, where $D_\varepsilon \Subset D$ is a sufficiently small open neigborhood of 0 in D. $\mathrm{Graph}(f)$ is clearly contained in the common zero set of the equations (1).

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In order to obtain the algebraic extension of f beyond the boundary of BSDs, one needs to make use of the fact that the Bergman kernel of any BSD Ω is a rational function; $K_\Omega(z, w) = \frac{1}{\mathrm{Vol}_{\mathrm{Euc}}(\Omega)} \cdot \frac{1}{h_\Omega(z,w)^{p(\Omega)+2}}$ when Ω is irr., as mentioned before. We note that K_Ω is simply the product of K_{Ω_j}, $1 \le j \le q$, for any reducible BSD $\Omega = \Omega_1 \times \cdots \times \Omega_q$, where Ω_j's are irr. BSDs.

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In 2012, N. Mok proved that if all irr. factors of a BSD D are of rank ≥ 2, then any holo. isometry $F : (D, \lambda ds_D^2) \to (\Omega, ds_\Omega^2)$, $\lambda > 0$, between BSDs D and Ω is totally geodesic, i.e., $(F(D), ds_\Omega^2|_{F(D)}) \subset (\Omega, ds_\Omega^2)$ is totally geodesic.

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(**Rank**-1 **case**) Let $F = (F_1, \ldots, F_m)$: $U \to \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$ be a holomorphic map from a connected open subset $U \subset \mathbb{B}^n$, $n \geq 2$, such that

$$\lambda(z, \overline{z}) ds_{\mathbb{B}^n}^2 = \sum_{j=1}^{m} \lambda_j(z, \overline{z}) F_j^*(ds_{\mathbb{B}^{N_j}}^2),$$

where $\lambda(z, \overline{z}), \lambda_j(z, \overline{z})$, $1 \leq j \leq m$, are positive smooth Nash algebraic functions on $\mathbb{C}^n$. Then, Y. Yuan and Y. Zhang (*J. Diff. Geom.* 2012) have proven that for each j, $1 \leq j \leq m$, either F_j is a constant map or F_j extends to a totally geodesic holomorphic isometric embedding from $(\mathbb{B}^n, ds_{\mathbb{B}^n}^2)$ to $(\mathbb{B}^{N_j}, ds_{\mathbb{B}^{N_j}}^2)$. Moreover, we have $\sum_{j \in J_0} \lambda_j = \lambda$, where $J_0 := \{j : 1 \leq j \leq m,\ F_j \text{ is not a constant map}\}$. This extends a rigidity result of N. Mok in 2002.

In 2016, N. Mok constructed holo. isometries $F_0 : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ (for some $n \geq 2$) into any irr. BSD Ω of rank ≥ 2 that are not totally geodesic.

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For BSDs D and Ω, we say that two holomorphic isometries f and g from $(D, \lambda ds_D^2)$, $\lambda > 0$, to (Ω, ds_Ω^2) are *equivalent* if $f = \Phi \circ g \circ \psi$ for some $\psi \in \mathrm{Aut}(D)$ and $\Phi \in \mathrm{Aut}(\Omega)$; otherwise, f and g are said to be *inequivalent*.

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$$f(\tau) := (\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}}), \quad \tau \in \mathcal{H},$$

where $\gamma := e^{\frac{\pi i}{p}}$ and $\tau^{\frac{1}{p}} = r^{\frac{1}{p}} e^{\frac{i\theta}{p}}$ for $\tau = re^{i\theta}$ with $r > 0$ and $0 < \theta < \pi$, and showed that $f : (\mathcal{H}, ds_{\mathcal{H}}^2) \to (\mathcal{H}, ds_{\mathcal{H}}^2)^p$ is a holo. isometric embedding that is not totally geodesic.

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Classification of holomorphic isometries $\Delta \to \Delta^p$

Around 2008, Sui-Chung Ng studied holomorphic isometries

$$F : (\Delta, k ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$$

for $k > 0$ and $p \geq 2$ during his Ph.D. studies at the University of Hong Kong (HKU). It's proved that $k \in \mathbb{Z}$ with $1 \leq k \leq p$.

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Example: Let $(\alpha_1, \alpha_2) : \Delta \to \Delta^2$ be the square root map with $\alpha_j(0) = 0$, $j = 1, 2$. From the functional equation we have

$$(1 - |\alpha_1(w)|^2)(1 - |\alpha_2(w)|^2) = 1 - |w|^2, \quad w \in \Delta.$$

Letting $f := (\alpha_1, \alpha_1 \circ \alpha_2, \alpha_2 \circ \alpha_2) : \Delta \to \Delta^3$, we have

$$(1 - |\alpha_1(z)|^2)(1 - |\alpha_1(\alpha_2(z))|^2)(1 - |\alpha_2(\alpha_2(z))|^2)$$
$$= (1 - |\alpha_1(z)|^2)(1 - |\alpha_2(z)|^2) = 1 - |z|^2$$

so that $f : (\Delta, ds_\Delta^2) \to (\Delta^3, ds_{\Delta^3}^2)$ is a holo. isometry. It's clear that $(\alpha_1, \alpha_2, \alpha_1, \alpha_2)$:$(\Delta, 2ds_\Delta^2) \to (\Delta^4, ds_{\Delta^4}^2)$ is a holo. isometry.
Question: Are all such holomorphic isometries F constructed from the q-th root maps for $2 \leq q \leq p$, and standard maps via compositions and combinations up to automorphisms of Δ and Δ^p? (Yes if $p = 2, 3$ or 4.)

Here, standard maps are those equivalent to one of the maps $z \mapsto (z, \mathbf{0})$, $z \mapsto (z, \ldots, z, \mathbf{0})$, $z \mapsto (z, \ldots, z)$.

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For any holo. isometry $F = (F_1, \ldots, F_p) : (\Delta, kds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$, Ng has introduced the so-called **sheeting numbers** $s_j \in \mathbb{Z}^+$, $1 \leq j \leq p$, for the component functions F_j, $1 \leq j \leq p$, and the **global sheeting number** n for F.

Classification of holomorphic isometries $\Delta \to \Delta^p$

For $p = 2, 3$, Ng has provided a positive answer to this question in his Ph.D. thesis (S.-C. Ng, *Proc. Amer. Math. Soc.* 2010). For $p = 2$, any holo. isometry $\Delta \to \Delta^2$ is equivalent to either the square root map or $z \mapsto (z, 0)$ when $k = 1$; $z \mapsto (z, z)$ when $k = 2$.

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For any holo. isometry $F = (F_1, \ldots, F_p) : (\Delta, kds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, Ng has introduced the so-called **sheeting numbers** $s_j \in \mathbb{Z}^+$, $1 \le j \le p$, for the component functions F_j, $1 \le j \le p$, and the **global sheeting number** n for F. Indeed, Ng obtained an irr. complex projective -algebraic curve $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ (resp. $V_j \subset \mathbb{P}^1 \times \mathbb{P}^1$) extending the graph of F (resp. F_j) and a finite branched covering map $V \to \mathbb{P}^1$ (resp. $V_j \to \mathbb{P}^1$) that extends the canonical projection onto the 1st factor, and n (resp. s_j) denotes the degree of the branched covering map, $1 \le j \le p$.

Rational functions

In addition, Ng (2010) has proven that from the construction of V_j, $1 \leq j \leq p$, we have a rational function $R_j : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$R_j(F_j(z)) = z \qquad \forall\, z \in \Delta$$

and

$$R_j\left(\frac{1}{\overline{z}}\right) = \frac{1}{\overline{R_j(z)}}, \qquad z \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1,$$

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To classify such holo. isometries $F = (F_1, \ldots, F_p) : (\Delta, k ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$, we study the behavior of R_j (resp. F_j) around the branch points together with the (polarized) functional equation.

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One of the important results of Ng (2010) is that if h is a component function of a holo. isometry from (Δ, kds^2_Δ) to $(\Delta^p, ds^2_{\Delta^p})$ such that h has exactly two branch points and its sheeting number is q, then h is a component function of the q-th root map.

Sheeting numbers

Moreover, these sheeting numbers satisfy the equality

$$\sum_{j=1}^{p} \frac{1}{s_j} = k \tag{2}$$

and $s_j | n$ for $j = 1, \ldots, p$; we also have $\frac{p}{k} \leq n \leq 2^{p-1}$.

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Difficulty in the general case: When $p = 5$, there are possible choices of s_j's satisfying (2) such that we couldn't find a holo. isometry constructed from the q-th root maps for $2 \leq q \leq 5$. This doesn't mean that such choices s_j's are coming from some holo. isometry $\Delta \to \Delta^5$, but we don't know how to rule out these possibilities. That's one reason why we still do not have a complete classification of all holo. isometries $\Delta \to \Delta^p$ for $p \geq 5$.

Theorem (C.-Mok, Math. Z. 2017)

Let $\Omega \Subset \mathbb{C}^N \subset X_c$ be the standard embeddings of an irreducible bounded symmetric domain Ω of rank ≥ 2 in its Harish-Chandra realization $\Omega \Subset \mathbb{C}^N$ as a bounded domain and its Borel embedding $\Omega \subset X_c$ as an open subset of its dual Hermitian symmetric space X_c. Let n be a positive integer, and $f : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ be a holomorphic isometric embedding. Denote by $\iota : X_c \hookrightarrow \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$ the minimal embedding of X_c defined by the positive generator $\mathcal{O}(1)$ of $\mathrm{Pic}(X_c) \cong \mathbb{Z}$. Then, $f(\mathbb{B}^n)$ is an irreducible component of some complex-analytic subvariety $\mathcal{V} \subseteq \Omega$ satisfying $\iota(\mathcal{V}) = P \cap \iota(\Omega)$, where P is some projective linear subspace of $\mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$.

Remark: Chan has realized the analogous theorem holds true for any holomorphic isometry $(\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, $p \geq 2$ in his Ph.D. thesis at HKU (2016).

Structure theorem for holomorphic isometries from $\mathbb{B}^n$

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It is still an open question on whether this theorem holds when $(\mathbb{B}^n, g_{\mathbb{B}^n})$ is replaced by $(\mathbb{B}^n, k g_{\mathbb{B}^n})$ for $3 \leq k < \mathrm{rank}(\Omega)$ and $\mathrm{rank}(\Omega) \geq 4$.

Here, the Harish-Chandra realization of a Hermitian symmetric space X_0 of the noncompact type means the Harish-Chandra embedding of X_0 in the complex Euclidean space $\mathbb{C}^N$ as a bounded domain (See Section 7 of Ch. VIII in the book written by S. Helgason, and the article of J. Wolf in 1972).

Minimal embedding of the compact dual X_c

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We now only work on irreducible BSD $\Omega \Subset \mathbb{C}^N$ and its compact dual Hermitian symmetric space X_c. In terms of the Harish-Chandra coordinates $z \in \mathbb{C}^N$, the minimal embedding (also called the first canonical embedding) $\iota : X_c \hookrightarrow \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$ may be written as

$$\iota(z) = [1, G_1(z), \ldots, G_{N'}(z)]$$

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Example: When $\Omega = D^I_{p,q}$ is the type-I classical domain ($1 \le p \le q$), then the compact dual $X_c = G(p,q)$ is the complex Grassmannian of complex p-planes in $\mathbb{C}^{p+q}$. The minimal embedding of $G(p,q)$ is the Plücker embedding $G(p,q) \hookrightarrow \mathbb{P}\left(\bigwedge^p \mathbb{C}^{p+q}\right) \cong \mathbb{P}^{\binom{p+q}{p}-1}$.

Lemma (C.-Mok, Math. Z. 2017)

Let $\Omega \Subset \mathbb{C}^N$ be an irr. BSD in its Harish-Chandra realization of rank r, then given any holomorphic isometry from $(\mathbb{B}^n, \lambda ds^2_{\mathbb{B}^n})$ to (Ω, ds^2_Ω), $\lambda > 0$, we have $\lambda = \frac{k(p(\Omega)+2)}{n+1}$ for some $k \in \mathbb{Z}$, $1 \le k \le r$.

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$$\omega_{X_c}|_{\mathbb{C}^N} = \sqrt{-1}\partial\overline{\partial}\log\left(1 + \sum_{l=1}^{N'} |G_l(z)|^2\right).$$

On the other hand, it is also known that $\omega_{X_c}|_{\mathbb{C}^N} = \sqrt{-1}\partial\overline{\partial}\log h_\Omega(z,-z)$ in terms of the Harish-Chandra coordinates $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$. Therefore, this implies $h_\Omega(z,\xi) = 1 + \sum_{l=1}^{N'} G_l(z)\overline{G_l(-\xi)}$.

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Note that O. Loos (1977) has provided a more precise description of these facts in his book, and one can also find related description of h_Ω in the study of bounded symmetric domains via the method of Jordan Triple systems.

On the other hand, we could obtain another expression (up to rescaling of the original Harish-Chandra coordinates $z = (z_1, \ldots, z_N)$) of $h_\Omega(z, z)$ as

$$\boxed{h_\Omega(z,z) = 1 - \sum_{j=1}^{N} z_j \overline{z_j} + \sum_{l=1}^{N''} (-1)^{\deg(\hat{G}_l)} \hat{G}_l(z) \overline{\hat{G}_l(z)}} \qquad (3)$$

for some homogeneous polynomial $\hat{G}_l(z)$ in z of degree ≥ 2 (up to $r := \mathrm{rank}(\Omega)$) for $1 \leq l \leq N''$. By polarization, we also have the expression of $h_\Omega(z, w)$ from (3), called the *generic norm* (O. Loos 1977).

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This is done by the fact that $\Omega \Subset \mathbb{C}^N$ is a bounded complete circular domain (which is not biholomorphic to the complex unit ball $\mathbb{B}^N$) with the use of the Bergman kernel, and that the original Harish-Chandra coordinates $z = (z_1, \ldots, z_N)$ are complex geodesic coordinates of Ω at $\mathbf{0}$ (or X_c at o) up to rescaling, meaning that $\zeta = (b_1 z_1, \ldots, b_N z_N)$ are complex geodesic coordinates for some real constant $b_j > 0$, $1 \leq j \leq N$.

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Now, the key idea in the proof of our theorem is to combine the relation between $h_\Omega(z,z)$ and the minimal embedding ι of X_c in $\mathbb{P}^{N'}$ (restricted to the open dense subset $\mathbb{C}^N \cong U_0 \subset X_c$), and the technique by N. Mok (2012) on the system of polarized functional equations. One shows that all defining equations of the variety extending the image of the holomorphic isometry are coming from linear equations on $\mathbb{P}^{N'}$. This yields our structure theorem.

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From the structure theorem of C.-Mok (*Math. Z.* 2017), the question naturally arises as to which linear sections $Z = \Lambda \cap \Omega$ are actually images of holomorphic isometries of complex unit balls, where Ω is an irr. BSD of rank ≥ 2. We study the particular case where $\Omega = D_n^{IV}$ is the type-*IV* classical domain whose compact dual is the hyperquadric $Q^n \subset \mathbb{P}^{n+1}$ for $n \geq 3$, i.e.,

$$Q^n = \{[z_1, \ldots, z_{n+1}, z_{n+2}] \in \mathbb{P}^{n+1} : \textstyle\sum_{j=1}^n z_j^2 - 2z_{n+1}z_{n+2} = 0\}.$$

Then, the embedding $\mathbb{C}^n \hookrightarrow Q^n \subset \mathbb{P}^{n+1}$ is given by

$$(z_1, \ldots, z_n) \to \left[z_1, \ldots, z_n, 1, \frac{1}{2}\sum_{j=1}^n z_j^2\right].$$

Let m and n be integers satisfying $m \geq 1$ and $n \geq 3$. We define

$$D_n^{IV} := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 < 2, \ \sum_{j=1}^{n} |z_j|^2 < 1 + \left| \frac{1}{2} \sum_{j=1}^{n} z_j^2 \right|^2 \right\}.$$

Note that the Kähler form corresponding to the Bergman metric $ds^2_{D_n^{IV}}$ on D_n^{IV} is given by

$$\omega_{ds^2_{D_n^{IV}}} = -n\sqrt{-1}\partial\overline{\partial} \log \left(1 - \sum_{j=1}^{n} |z_j|^2 + \left| \frac{1}{2} \sum_{j=1}^{n} z_j^2 \right|^2 \right).$$

Moreover, the corresponding Kähler form of the Kähler metric $g_{D_n^{IV}}$ on D_n^{IV} is given by $\omega_{g_{D_n^{IV}}} = \frac{1}{n}\omega_{D_n^{IV}}$. We also have $p(D_n^{IV}) = n - 2$.

Holomorphic isometries from $\mathbb{B}^m$ to D_n^{IV} $(n \geq 3)$

Recall n, m are integers such that $m \geq 1$ and $n \geq 3$. We observe from the functional equations that up to equivalence the images of holo. isometries from $(\mathbb{B}^m, g_{\mathbb{B}^m})$ to $(D_n^{IV}, g_{D_n^{IV}})$ can be extended to some affine-algebraic subvarieties $\mathcal{V}_{\mathbf{A}'}$ in $\mathbb{C}^n$ which are defined on next page. More precisely, an irreducible component of the intersection $\Sigma_{\mathbf{A}'} = \mathcal{V}_{\mathbf{A}'} \cap D_n^{\mathrm{IV}}$ would be the image of some holo. isometry F: $(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$, where $\mathbf{A}' \in M(n-m, n; \mathbb{C})$ is a matrix satisfying $\mathbf{A}'\overline{\mathbf{A}'}^T = \mathbf{I_{n-m}}$.

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In particular, we classify images of all holo. isometries from $(\mathbb{B}^m, g_{\mathbb{B}^m})$ to $(D_n^{IV}, g_{D_n^{IV}})$ for integers $n \geq 3$, $m \geq 1$. Note that given a holo. isometry from $(\mathbb{B}^m, kg_{\mathbb{B}^m})$ to $(D_n^{IV}, g_{D_n^{IV}})$ for some positive integer k, we have $k = 1$ or 2. We show that if there is a holo. isometry from $(\mathbb{B}^m, kg_{\mathbb{B}^m})$ to $(D_n^{IV}, g_{D_n^{IV}})$ with $m \geq 2$ and $n \geq 3$, then $k = 1$. Thus, for any holo, isometry $f : (\mathbb{B}^m, 2g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$, we have $m = 1$, and f is totally geodesic by the Gauss equation.

Affine-algebraic subvarieties $\mathcal{V}_{\mathbf{A}'}$ of $\mathbb{C}^n$

For $1 \leq m \leq n-1$ and $n \geq 3$, let $\mathbf{A}' \in M(n-m, n; \mathbb{C})$ be a matrix of rank $n-m$. If $1 \leq m \leq n-2$, then we let

$$\mathcal{V}_{\mathbf{A}'} := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \mathbf{A}' \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \sum_{j=1}^n z_j^2 \\ \mathbf{0}_{(n-m-1)\times 1} \end{pmatrix} \right\}.$$

If $m = n-1$, then $\mathbf{A}' = \mathbf{v} = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \in M(1, n; \mathbb{C})$ and we let $\mathcal{V}_{\mathbf{v}} \subseteq \mathbb{C}^n$ be the affine-algebraic subvariety defined by $\sum_{j=1}^n v_j z_j - \frac{1}{2} \sum_{j=1}^n z_j^2 = 0$. Moreover, we define

$$\Sigma_{\mathbf{A}'} := \mathcal{V}_{\mathbf{A}'} \cap D_n^{IV}.$$

Existence Theorem

Theorem (C.-Mok, Math. Z. 2017)

Let n and m be integers satisfying $1 \leq m \leq n-1$ and $n \geq 3$. Let $\mathbf{A}' \in M(n-m, n; \mathbb{C})$ be a matrix satisfying $\mathbf{A}'\overline{\mathbf{A}'}^T = \mathbf{I_{n-m}}$. Then, the irreducible component $\widetilde{W}$ of $\Sigma_{\mathbf{A}'}$ containing $\mathbf{0}$ is the image of some holomorphic isometry $F\colon (\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$.

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Outline of the proof: We make use of the fact that there is a matrix $\mathbf{U}' \in M(m, n; \mathbb{C})$ such that $\begin{bmatrix} \mathbf{U}' \\ \mathbf{A}' \end{bmatrix} \in U(n)$ is an $n \times n$ unitary matrix and the restriction of the local Kähler potential of $(D_n^{IV}, g_{D_n^{IV}})$ to the germ of $\Sigma_{\mathbf{A}'}$ at $\mathbf{0}$ is equivalent to that of $(\mathbb{B}^m, g_{\mathbb{B}^m})$. We actually show that $\Sigma_{\mathbf{A}'}$ is a smooth m-dimensional complex-analytic subvariety of D_n^{IV}. Then, $(\widetilde{W}, g_{D_n^{IV}}|_{\widetilde{W}})$ is locally holomorphically isometric to $(\mathbb{B}^m, g_{\mathbb{B}^m})$ and the result follows from the extension theorem of Mok (2012). $\square$

Uniqueness Theorem

Theorem (C.-Mok, Math. Z. 2017)

Let $F : (\mathbb{B}^m, \lambda ds^2_{\mathbb{B}^m}) \to (D^{IV}_n, ds^2_{D^{IV}_n})$ *be a holomorphic isometry, where* $n \geq 3$ *and* $m \geq 1$ *are integers, and* $\lambda > 0$ *is a real constant. Then, either* $\lambda = \frac{n}{m+1}$ *or* $\lambda = \frac{2n}{m+1}$ *and we have the following:*

1. *If* $\lambda = \frac{n}{m+1}$*, then* $1 \leq m \leq n-1$ *and* $F = \widetilde{f} \circ \rho$ *for some holomorphic isometry* $\widetilde{f} : (\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D^{IV}_n, g_{D^{IV}_n})$ *and some (totally geodesic) holomorphic isometry* $\rho : (\mathbb{B}^m, g_{\mathbb{B}^m}) \to (\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}})$.
2. *If* $\lambda = \frac{n}{m+1}$ *and* $m = n-1$*, then* F *is equivalent to the non-standard holomorphic isometry constructed by Mok (2016)* $\iff$ $\exists\, \Psi \in \mathrm{Aut}(D^{IV}_n)$ *such that* $\Psi(F(\mathbf{0})) = \mathbf{0}$ *and* $\Psi(F(\mathbb{B}^{n-1}))$ *is the irreducible component of some complex-analytic subvariety* $\Sigma_{\mathbf{c}} \subset D^{IV}_n$ *containing* $\mathbf{0}$ *for some* $\mathbf{c} \in M(1, n; \mathbb{C})$ *satisfying* $\mathbf{c}\overline{\mathbf{c}}^T = 1$ *and* $\mathbf{c}\mathbf{c}^T = 0$.

As mentioned before, if $\lambda = \frac{2n}{m+1}$, then $m = 1$ and F is totally geodesic.

Remarks

From Y. Zhang's study on sub-VMRT structure modeled on (Q^m, Q^n) (arXiv:1503.05284), we also obtained new examples of holomorphic isometries from $(\mathbb{B}^m, g_{\mathbb{B}^m})$ into $(D_n^{IV}, g_{D_n^{IV}})$ with explicit parametrizations for $1 \leq m \leq n-1$ and $n \geq 3$. When $m = n-1$, one particular example is given by

$$f(w_1, \ldots, w_{n-1}) = \left(w_1, \ldots, w_{n-1}, 1 - \sqrt{1 - \sum_{j=1}^{n-1} w_j^2} \right).$$

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From item 2 in the previous theorem, we have actually showed that there are holo. isometries from $(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}})$ to $(D_n^{IV}, g_{D_n^{IV}})$ which are inequivalent to those constructed by Mok (2016). The above map f is a particular example. This answers the question raised by N. Mok (around 2016) about the uniqueness of holo. isometries from $\mathbb{B}^{p(\Omega)+1}$ to an irr. BSD Ω of rank ≥ 2 when the target Ω is the type-IV classical domain D_n^{IV}, $n \ge 3$.

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Remarks

There are articles of Xiao-Yuan (*J. Math. Pures Appl.* 2020) and Upmeier-Wang-Zhang (*Int. Math. Res. Not. IMRN* 2019) studying holomorphic isometries of the complex unit ball into bounded symmetric domains. Particularly, both articles gave explicit parametrizations of all holomorphic isometries of $\mathbb{B}^{n-1}$ into D_n^{IV} for any integer $n \geq 3$. Note that Xiao-Yuan has also obtained the map f by using the functional equation and doing reparametrizations.

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The existence of holomorphic isometries $(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D_n^{IV}, g_{D_n^{IV}})$, $n \geq 3$, which are inequivalent to those constructed by Mok (2016) was also obtained independently by Xiao-Yuan (*J. Math. Pures Appl.* 2020) and Upmeier-Wang-Zhang (*Int. Math. Res. Not. IMRN* 2019).
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There are further developments in the study of holomorphic isometries between BSDs by C.-Yuan (*Ann. Inst. Fourier (Grenoble)* 2019), and Ming Xiao (*J. Reine Angew. Math.* **789** (2022)), etc.

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Thank you!