2.3. Intersection pairing and Picard-Lefschetz theory

Exact symplectic Lefschetz fibration

Setup. $\begin{array}{c} X \\ \downarrow \pi \\ \mathbb{C} \end{array}$ (ω, θ) $d\theta = \omega$, noncompact, with ∂

(i) π has only finitely many critical points

$\forall x \in \mathrm{Crit}(\pi)$, $\exists J$ s.t. compatible w/. ω, integrable near x
(locally defined near x)

s.t. $\exists$ $(z_1, \cdots, z_n)$ hol. coordinates, $\pi(z_1, \cdots, z_n) = z_1^2 + \cdots + z_n^2$

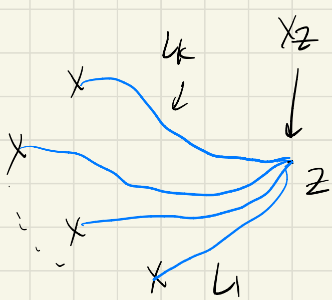
(ii) Away from $\mathrm{Crit}(\pi)$, π is a symplectic fibration

(iii) $\forall z$, regular value of π, $(X_z, \omega|_z, \theta|_z)$
$\Rightarrow$ compact Liouville domain

(iv). "Trivial" near the vertical ∂.

$\leftarrow$ Lefschetz thimbles

$\mathbb{C}$

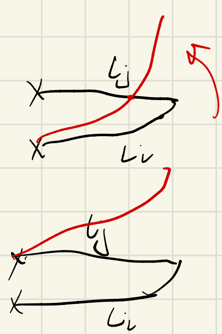


$\partial L_i = V_i \subseteq X_z$
Lag sphere
vanishing cycle.

non-symmetric pairing $\bigoplus_i \mathbb{Z} L_i$

$L_i \cdot L_j := \tilde{L}_i \cdot L_j$

$L_j \cdot L_i = \tilde{L}_j \cdot L_i = 0$.



Define. $A_{ij} = L_i \cdot L_j$ $\quad (A_{ij})_{k\times k}$

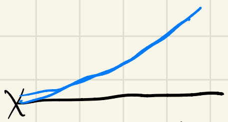
$B_{ij} = V_i \cdot V_j$ $\quad (B_{ij})_{k\times k}$.

(defined in the smooth fiber X_z).

$n \Rightarrow$ dimension of total space

Prop^n. $B = A - (-1)^n A^t$. $(*)$

proof. ① Look @ the diagonal entries, $i=j$.

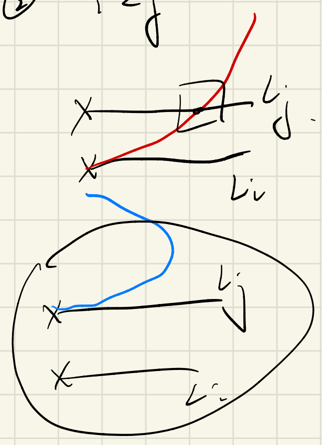


$\Rightarrow$ $L_i \cdot L_j = 1$

$V_i \Rightarrow$ sphere. S^{n-1} $n-1$

$\chi(S^{n-1}) = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$ $= 1-(-1)^n$

② $i<j$



$\Rightarrow$ $\boxed{L_i \cdot L_j = V_i \cdot V_j}$

$\underset{V_i \cdot V_j}{\underset{\|}{B_{ij}}} = \underset{L_i \cdot L_j}{\underset{\|}{A_{ij}}} - (-1)^n \underset{0}{\underset{\|}{A_{ji}}}$.

✓.

③ $i>j$, similar proof. $\square$.

Rmk. $[L_i] \Rightarrow$ define relative homology classes $H_n(X; X_z)$

$$\cdots \to \cancel{H_n(X_z)} \to H_n(X) \to H_n(X; X_z) \to H_{n-1}(X_z) \to \cdots$$

0

If $H_n(X_z) = 0$ (e.g. fibers are Weinstein domains). $\Rightarrow$ one can study the intersection pairing by dim reduction

$\Rightarrow$ For any $Y \hookrightarrow X$, closed, oriented, $[Y] \in H_n(X; X_z)$, $[Y] = \sum a_j [L_j]$

$$\begin{array}{c} X \\ \downarrow \pi \\ \mathbb{C} \end{array}$$

Def$^{\underline{n}}$. Monodromy of the fiber: parallel transport along the blue circle.

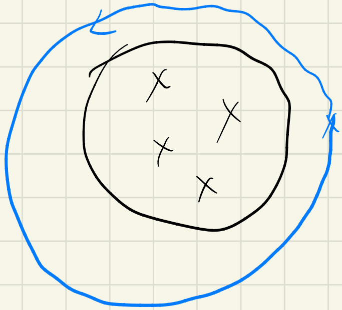
symplectic

Global

Exercise. The Hamiltonian isotopy class of the monodromy is independent of the choice of the loop.

$\varphi : F = X_z \to F$.

$\mathbb{C}$



Example. $\begin{array}{c} \mathbb{C}^n \\ \downarrow z_1^2+\cdots+z_n^2 \\ \mathbb{C} \end{array}$ $\Rightarrow$ the monodromy: $T^*S^{n-1} \to T^*S^{n-1}$ is called the Dehn twist along S^{n-1}.

cotangent

$(T^*S^{n-1}, \sum dp_i \wedge dq_i)$ Choose a Riemannian metric.

Consider the Hamiltonian $(q,p) \mapsto \|p\| = H$

$\Rightarrow$ smooth away from the 0-section.

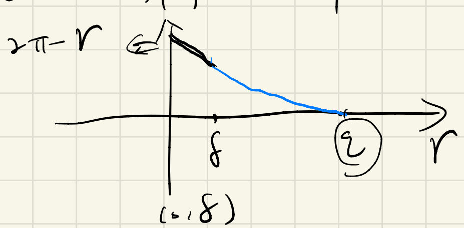
Fact. Hamiltonian flow of H $T^*S^{n-1} \setminus S^{n-1}$ is the same as the normalized geodesic flow.

(q,p)

S^{n-1}

$\varepsilon > 0$

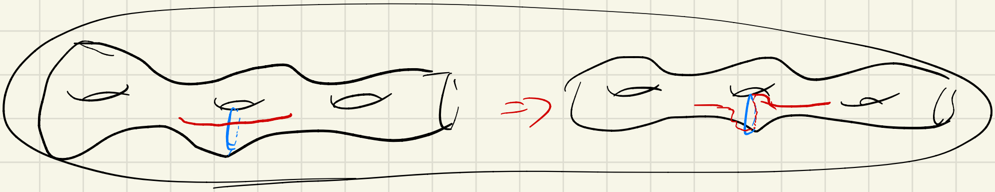
⊙ A cut-off function: $\psi_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ $\quad 0 < \delta \ll \varepsilon$



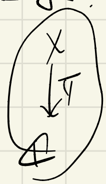
Def$^{\underline{n}}$. (model Dehn twist).

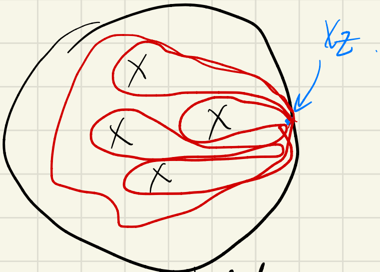
$$\tau_{S^{n-1}}(q,p) := \begin{cases} \text{for } p \neq 0, & \phi^H_{\psi_\varepsilon(\|p\|)}(q,p) \\ \text{for } p = 0, & \text{anti-podal map } p \mapsto -p. \end{cases}$$

Exercise. smooth, symplectic, Ham isotopic to the monodromy of the standard model.



E.g.





$T^*S^{n-1} \circlearrowleft \tau_{S^{n-1}}$ standard

vanishing cycles $V_1, \dots, V_k$

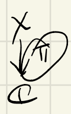
Claim. $\varphi = \tau_{V_1} \circ \tau_{V_2} \circ \dots \circ \tau_{V_k}$ Hamiltonian isotopic to

Exercise: Make it precise!

If 'Lagrangian embedding' $\boxed{S^{n-1} \hookrightarrow (X_z, \omega_z, \theta_z)}$ V.

Weinstein tubular nbhd theorem $\Rightarrow \exists\ (D^*_\epsilon S^{n-1}, \Sigma\, dp \wedge dq) \xhookrightarrow{\text{symp}} (X_z, \omega_z)$

τ_V := plugging in the model Dehn twist and extend by identity outside $D^*_\epsilon S^{n-1}$.

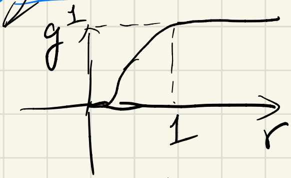


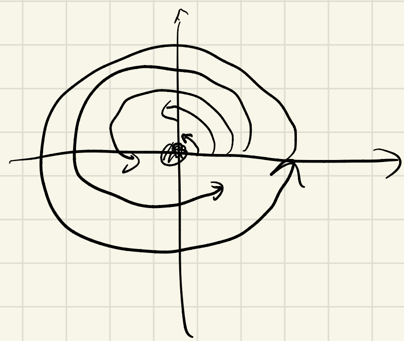
Monodromy of the total space

$\mathbb{C}_z$

$\frac{i}{2} dz \wedge d\bar{z}$

$H := g(|z|)\ \boxed{\pi |z|^2}$ $\Rightarrow$ rotate by 2π

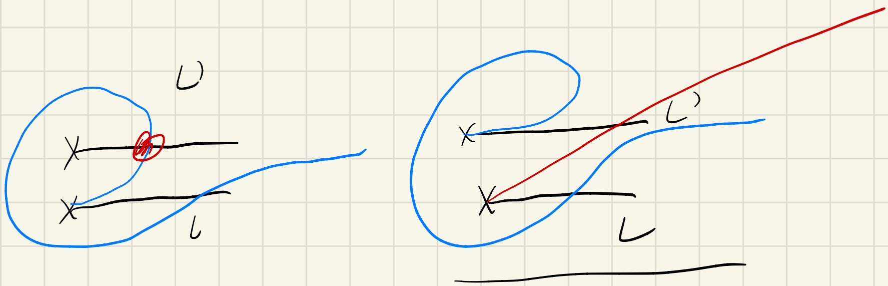




Def^n. is defined to be the time-1 Ham diff of H.

(change z to $\pi(x)$) $\Rightarrow \phi^1$

observations



Look@ $L' \cdot \phi'(L) = V' \cdot V \cdot (-1)^n$

$L \cdot \phi'(L') = 0$.

Prop^n. $L_i \cdot \phi'(L_j) = (-1)^n L_j \cdot L_i$.

proof: By above picture. □

$H_n(X;F) \circlearrowleft (\phi^1)_*$ N under the basis $L_1, \dots, L_k$

Claim. $N = (-1)^n A^{-1} A^t$

Exercise. Prove it using the above observations.

- Smooth fiber $\Rightarrow T^*S^{n-1}$
- vanishing cycles $\Rightarrow$ 0-section

observation $\phi^1 : X \to X$ preserves the smooth fiber F near ∞.

acts by ψ^1 (global monodromy of the fiber).

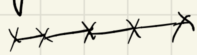
Exercise. $A_k := \{p(y) + z_1^2 + \dots + z_n^2 = 0\} \subseteq \mathbb{C}^{n+1}_{y, z_1, \dots, z_n}$

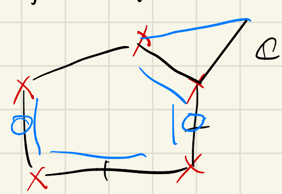
$p(y) \neq 0$

$p(y)$: polynomial of degree $k+1$

$\downarrow y$

$\mathbb{C}$





(1) This is a Lefschetz fibration (Morse type singularities) if and only if $p(y)$ only has simple roots. critical values $\Leftrightarrow$ roots of p.

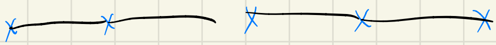
(2) Show that $H_n(A_k; \mathbb{Z})$ has rank k.

generated by matching cycles: glue the Lefschetz thimbles together along their boundary.

(3) compute A, B, N

(4) Can deform the $\omega_{std}^{n+1}\big|_{A_k}$ s.t. . . . — defines a symplectic Lefschetz fibration.

(Khovanov-Seidel, ~2002, J.A.M.S)



3. Fukaya–Seidel category & Floer theory.

$$F \to X \quad \downarrow \pi \quad \mathbb{C}$$

$L_1, \cdots, L_k$ Lefschetz thimbles

$V_1, \cdots, V_k$ vanishing cycles

(1) $\begin{cases} L_i \cdot L_j = \underline{V_i \cdot V_j} & i<j \\ 1 & i=j \\ 0 & i>j \end{cases}$

$(A_{ij})_{k\times k}$ upper triangular with diagonal entries 1.

(2) $\phi^1 : X \to X$ restricts to $\varphi^1 : F \to F$

(3) $(\phi^1)_* : H_n(X;F) \to H_n(X;F)$, $B = A - (-1)^n A^t$.
$\quad\;\; \shortparallel$
$\quad (-1)^n A^{-1} A^t$

(4) $\underline{L_i \to \partial L_i = V_i \text{ . restriction map}}$.

<u>Categorify all of these!</u>

L, L' Lagrangians $\Rightarrow$ Floer cohomology $HF^*(L, L')$

$L \cdot L'$ $\qquad \chi(HF^*(L,L')) = L \cdot L'$

(1) $X \xrightarrow{\pi} \mathbb{C} \rightsquigarrow$ Fukaya–Seidel category. $\mathcal{F}(\pi) = \mathcal{A}$ $\quad$ k-linear category. (k ← field)

$\deg(x)$ = orientation sign. (hom(-,-) are k-vector spaces).

objects: $L_1, \cdots, L_k$

<u>morphism spaces</u> $\hom(L_i, L_j) := \begin{cases} \bigoplus_{x \in V_i \cap V_j} k \cdot x & i<j \\ k \cdot e_{L_i} & i=j \\ 0 & i>j \end{cases}$

<u>A_∞-category</u>. $\mu^k : \hom(L_{i_{k-1}}, L_{i_k}) \otimes_k \cdots \otimes_k \hom(L_{i_0}, L_{i_1}) \to \hom(L_{i_0}, L_{i_k})$.

$(\hom(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \hom(L_{i_{k-1}}, L_{i_k}))$.

<u>quadratic relation</u>: $\mu^k(\cdots, \mu^*(\cdots), \cdots) = 0$.

$k=1 \qquad \overset{\Downarrow}{(\mu^1)^2} = 0 \qquad (\hom(L, L'), \mu^1) \Rightarrow$ chain complex

(Floer cochain complex)

(2) monodromy

$\psi' : F \to F$

$\varphi' : X \to X$

$\mathrm{Lef}(\psi') := \mathrm{str}\left(\psi'_* : H_*(F;k) \to H_*(F;k)\right)$

$\Delta \cdot \mathrm{graph}(\psi') = \mathrm{Lef}(\psi')$

symplectically: fixed point Floer cohomology. $HF^*(\psi') = (CF^*(\psi'), d)$.

assume: ψ' ⇒ compactly supported

$\forall x$, s.t. $\psi'(x) = x$, $\det(D_x\psi' - Id) \neq 0$.

⇒ $Fix(\psi')$ is a finite set.

$\mathbb{Z}/2$-graded

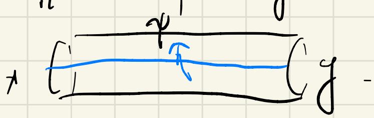
Dostoglou–Salamon — 1994 Annals "Atiyah–Floer"

$$CF^*(\psi') := \bigoplus_{x \in Fix(\psi')} kx$$

$\deg(x) = \mathrm{sign}(\det(D_x\psi - Id))$.

3-mfld: mapping torus (Σ, Φ).

differential: counting "twisted" holo. cylinders.



$\partial_s u + J_t(u)\partial_t u = 0$

$J_t : \mathbb{R} \to End(TF)$ $\quad J_t^2 = -id$

$\psi_*(J_t) = J_{t+1}$

asymptotic to x, y on the ends

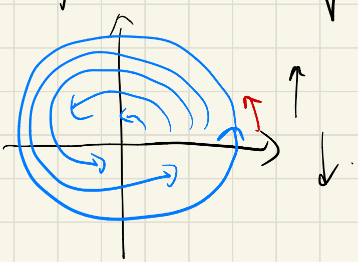
$\langle dx, y \rangle := \#\overline{\mathcal{M}}(x,y)/\mathbb{R}$.

$\Phi \Rightarrow \mathcal{M}(\Sigma; L)$

moduli space stable $SO(3)$ bundle

$\phi' : X \to X$.

To define Floer theory, need to remove the fixed points near ∞.



Solution: compose ϕ' with

① Ham diff associated with $\varepsilon\pi|z|^2$ → $HF^*(\phi^{+\varepsilon})$

② use translation symmetry → $HF^*(X;1)$.

i.e. Ham diff of

$A \cdot \mathrm{Re}(z)$

<u>Prop^n</u>. $\exists$ long exact sequence

$$\cdots \to HF^*(X;1) \to HF^*(\varphi^{1+\varepsilon}) \to HF^{*-1}(\varphi) \to \cdots$$

Actually, can consider $HF^*(\varphi^r)$

$HF^*(X;r) \to$ use translation $\pi \cdot r|z|^2$

$HF^*(\varphi^{r+\varepsilon}) \to$ use rotation

and $\exists$ long exact sequences

$$\cdots \to HF^*(X;r) \to HF^*(\varphi^{r+\varepsilon}) \to HF^{*-1}(\varphi^r) \to \cdots$$

$$\cdots \to HF^*(\varphi^{r+\varepsilon}) \to HF^*(X;r+1) \to HF^{*-1}(\varphi^{r+1}) \to \cdots$$

(3) Why are the L.E.S. useful?

If $F \circlearrowleft \Phi$, s.t. $\Phi = \tau_{V_1} \circ \cdots \circ \tau_{V_k}$

want to ask the growth behavior of # of fixed points of Φ^r.

By the L.E.S. $\Rightarrow$ we can "recover" $HF^*(\Phi^r)$ from $HF^*(X;r)$, $HF^*(\varphi^{r+\varepsilon})$ etc.

<u>Fact</u>. $F \Rightarrow$ Liouville domain, $V_1, \ldots, V_k$ Lag spheres

Then exists $X \xrightarrow{\pi} \mathbb{C}$, s.t. the vanishing cycles are given by $\{V_1, \ldots, V_k\}$.

$D^b Coh(X)$

(4) Can understand $HF^*(X;r)$, $HF^*(\varphi^{r+\varepsilon})$ from the $\mathcal{F}(\pi)$.

① $\mathcal{F}(\pi)$ is proper and homologically smooth.

proper: $\forall L, L'$, $H^*(\hom(L,L'))$ is finite-dimensional.

homologically smooth: Δ-bimod is perfect. (split-generated by Yoneda bimodules)

② The induced action of $\varphi': \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ coincides with the Serre functor $S: \mathcal{F}(\pi) \to \mathcal{F}(\pi)$. $D^b(\mathrm{Coh}(X))$

$\mathcal{A}(S(L), L') = \mathcal{A}(L', L)^\vee$. $S = \omega_X[n] \otimes$

③ fixed point theory on $\mathcal{F}(\pi)$

"twisted Hochschild homology" $HH_*(\mathcal{A}, S^{-1}) \cong HH_*(\mathcal{A}, \varphi')$.

A_∞-functor (A_∞-bimodule). $HH_*(\mathcal{A}; S^{-r}) \cong HH_*(\mathcal{A}, (\varphi')^r)$.

④ $\exists$ twisted open-closed string map

$$\mathcal{OC}_r: HH_*(\mathcal{A}; S^{-r}) \to HF^{*+n}(X; r).$$

Conj. (Seidel). $\mathcal{OC}_r$ is an isomorphism.

Thm (B-Seidel). $\mathcal{OC}_r$ is an injection $\Rightarrow \mathrm{rank}_k HH_*(\mathcal{A}; S^{-r}) \le \mathrm{rank}_k HF^{*+n}(X; r)$.

(*)

$(-1) A^{+} A^{e}$ $\begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$

Thm. (F, ω_F, θ_F) Liouville domain

(V_0, V_1) Lagrangian spheres, s.t. $V_0 \pitchfork V_1$, $HF^*(V_0, V_1)$ has rank at least 2.

Then $\varphi := \tau_{V_0} \circ \tau_{V_1}$, $\#\mathrm{Fix}(\varphi^r)$ grows exponentially fast as $r \to \infty$.

why? Step 0. $\exists$ $X \xrightarrow{\pi} \mathbb{C}$, smooth fiber $\simeq F$, has 2 vanishing cycles V_0, V_1

Step 1. By L.E.S. $\Rightarrow$ rank $HF^*(\varphi^r)$ grows exponentially iff $HF^*(X; r)$ grows exponentially. rank of $HF^*(X; \varphi^{r(\epsilon)})$

$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Step 2. Thm (*) if rank $HH_*(\mathcal{A}; S^{-r})$ grows ...,
then rank of $HF^*(X, r)$ grows exp'ly ....

Step 3. Algebra. □