Lefschetz fibrations and their applications in symplectic topology.

1. Lefschetz pencils in algebraic geometry
2. Symplectic version: vanishing cycles, Lefschetz thimbles, Picard-Lefschetz theory
3. <u>Thm</u>. (B-Seidel)

   (X, ω) is a Liouville domain (or aspherical closed symplectic mfld),

   V_0, V_1 Lagrangian spheres $V_0, V_1 \hookrightarrow X$ $\quad V_0 \neq V_1$

rank$_{\mathbb{C}}$ $HF^*(V_0, V_1) \geq 2$ (minimal # of $V_0 \pitchfork V_1 \geq 2$)

$\Rightarrow \phi = \tau_{V_0} \circ \tau_{V_1}$ (composition of Dehn twists),

then # of fixed points ϕ^r grow exponentially fast as $r \to \infty$.

<u>Setup</u>. $X \Rightarrow$ smooth algebraic variety over $\mathbb{C}$ (complex mfld).

<u>Def^n</u>. $f: X \to \mathbb{C}$ holomorphic function (regular) over X

x is a critical point, i.e. $df(x) = 0$

$$\boxed{df = \Sigma \frac{\partial f}{\partial z_i} dz_i}$$

x is non-degenerate if

$$\begin{tikzcd} T^*X \arrow[d] \\ X \arrow[u, bend right, "df"'] \end{tikzcd}$$

x is a transverse intersection between graph(df) and the 0-section.

Exercise. Show that $x \in \mathrm{Crit}(f)$ non-degenerate
if $\mathrm{Hess}(f)^{(x)} := \left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right)(x)$ is
a non-singular matrix.

$(z_1, \cdots, z_n)$
local hol. coordinates
of X

Exercise. Independence on the choices of local hol. coordinates.

Holomorphic Morse Lemma.

Suppose $x \in \mathrm{Crit}(f)$ is non-degenerate. Then $\exists$ local hol. coordinates $(z_1, \cdots, z_n)$, s.t. $\boxed{f = f(0) + z_1^2 + \cdots + z_n^2}$.

($x \mapsto 0$ under these coordinates)

Rmk. In the real case, $f = f(0) + \underbrace{x_1^2 + \cdots + x_k^2} - (x_{k+1}^2 + \cdots + x_n^2)$.

proof. Proof by induction.

For $n=1 \Rightarrow$ trivial. (normalize).

Assume that the lemma holds for all smooth complex mfld of $\dim_{\mathbb{C}} \le n-1$.

Need to prove the statement for $\dim_{\mathbb{C}} X = n$.

$x \in X$, choose a smooth hypersurface $Y \subseteq X$
passing through x,
locally defined by $\{t=0\}$. $dt \neq 0$ at x.

Now apply induction hypothesis, consider
$f|_Y$, then $x \in \mathrm{Crit}(f|_Y)$.

$\Rightarrow \exists$ holomorphic coordinates $z_1, \cdots, z_{n-1}$ of Y, s.t. $f|_Y = f(0) + z_1^2 + \cdots + z_{n-1}^2$.

Extend $z_1, \cdots, z_{n-1}$ to hol. functions on X,
$\boxed{f = f(0) + z_1^2 + \cdots + z_{n-1}^2 + t g}$

$f = f(0) + z_1^2 + \cdots + z_{n-1}^2 + t\, g$

$(z_1, \cdots, z_{n-1}, t) \Rightarrow$ local coordinates near x.

Because $x \in \mathrm{Crit}(f)$, $g(0) = 0$.

We can write $g = 2\sum_{i=1}^{n-1} \alpha_i z_i + t\phi$.

$$f = f(0) + z_1^2 + \cdots + z_{n-1}^2 + \boxed{2\alpha_1 z_1 t} + \cdots + 2\alpha_{n-1} z_{n-1} t + t^2\phi$$
$$= f(0) + (z_1 + \alpha_1 t)^2 + \cdots + (z_{n-1} + \alpha_{n-1} t)^2 + t^2(\phi - \alpha_1^2 - \cdots - \alpha_{n-1}^2).$$

Because x is non-degenerate, $\psi = \phi - \alpha_1^2 - \cdots - \alpha_{n-1}^2$ satisfies $\boxed{\psi(0) \neq 0}$.

Define $z_1' = z_1 + \alpha_1 t, \cdots, z_{n-1}' = z_{n-1} + \alpha_{n-1} t$, $\boxed{z_n^2 = t^2\psi}$. $\square$

<u>Defⁿ</u>. (Pencils of hypersurfaces). $X \Rightarrow$ compact complex mfld (smooth projective variety over $\mathbb{C}$).

$\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ is a holomorphic line bundle.

Pencil of hypersurfaces in $X \Longleftrightarrow \mathbb{P}^1 \hookrightarrow \boxed{\mathbb{P}(H^0(X;L))}$

$H^0(X;L)$ ⇓ $\mathbb{C}$-vector space of global hol. sections.

i.e. a pair sections σ_0, σ_∞ of $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$, $\mathbb{P}^1 \hookrightarrow \mathbb{P}(H^0(X;L))$ is defined by $t \mapsto \sigma_0 + t\cdot\sigma_\infty$.

B (base locus) $:= \sigma_0^{-1}(0) \cap \sigma_\infty^{-1}(0)$.

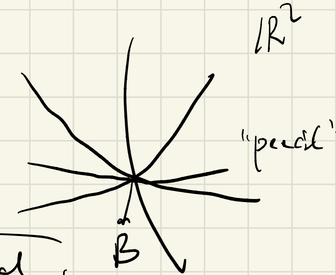
<u>Defⁿ</u>. (Lefschetz pencil) $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ $\sigma_0, \sigma_\infty \Rightarrow$ pencil is called Lefschetz if

① B is smooth of codim 2.

② $\begin{matrix} X \setminus B \\ \downarrow \\ \mathbb{P}^1 \end{matrix}$ $\begin{matrix} x \\ \downarrow \\ [\sigma_0(x) : \sigma_\infty(x)] \end{matrix}$ has only non-degenerate critical points (locally modeled on $(z_1, \cdots, z_n) \mapsto z_1^2 + \cdots + z_n^2$)

Exercise $(\mathbb{P}^2, \mathcal{O}(3))$ $\qquad \sigma_0 = x_0^3 + x_1^3 + x_2^3$

(Fermat pencil) $\qquad \overset{\shortparallel}{K_{\mathbb{P}^2}^{-1}} \qquad \sigma_\infty = x_0 x_1 x_2$



Prove this defines a Lefschetz pencil.

Prop^n $\quad \begin{array}{c} L \\ \downarrow \\ X \end{array} \quad s_0, \dots, s_N \in H^0(X;L)$ basis of global sections

assume that $\iota : X \longrightarrow \mathbb{P}^N$

$$x \longmapsto [s_0(x) : \cdots : s_N(x)]$$

defines a holomorphic embedding.

Then for $\mathbb{P}^1 \hookrightarrow \mathbb{P}(H^0(X;L))$ $\boxed{\text{generic}}$,

the corresponding pencil is a Lefschetz pencil.

proof. Note that
$$\begin{array}{ccc} L & \longrightarrow & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}^N \end{array}$$
is commutative. $\qquad \iota^* \mathcal{O}(1) \cong L$.

Then $\mathbb{P}(H^0(X;L)) \cong \mathbb{P}(H^0(\mathbb{P}^N; \mathcal{O}(1))) \cong (\mathbb{P}^N)^\vee$

linear system associated with L

universal critical locus $\qquad Z \subseteq X \times (\mathbb{P}^N)^\vee \qquad n+N$

$$:= \left\{ (x, H) \;\middle|\; \begin{array}{l} x \in X_H \\ x \text{ is a singular point of } X_H \end{array} \right\}$$

$\sigma_H \in H^0(X;L)$

$X_H := \sigma_H^{-1}(0)$

Step 1. Z is a smooth algebraic variety of dimension $N-1$.

(implicit function theorem)

$(t_1, \dots, t_N) \hookrightarrow s_0 + t_1 s_1 + \cdots + t_N s_N$

$(z_1, \dots, z_n)$, $\quad H = s_0^{-1}(0)$

$(t_1, \dots, t_N)$

$(n+N) - (n+1) = N-1$

Exercise. Show that the derivative of $\;\;$ is surjective.

$$\begin{cases} s_0(x) + t_1 s_1(x) + \cdots + t_N s_N(x) = 0 \\ \dfrac{\partial}{\partial z_i} (s_0(x) + t_1 s_1(x) + \cdots + t_N s_N(x)) = 0, \quad i = 1, \dots, n \end{cases} \qquad n$$

Step 2. For $(x,H) \in Z$, $\begin{array}{c} L \\ \downarrow \\ X \end{array} \circlearrowright s_H$, s_H is non-degenerate

if and only if

$pr : Z \to \mathbb{P}^N$ is an immersion.

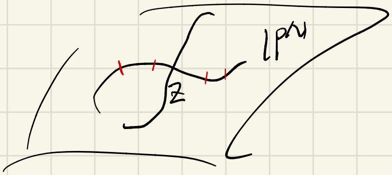
$\quad\; \cap$

$X \times \mathbb{P}^N$

(Exercise)

Step 3. Consider the projection $pr(Z) \subseteq \mathbb{P}^N$

$$\begin{cases} \text{codim} \geq 1 \\ \text{immersion is generic (non-immersed points has codim} \geq 1 \text{ inside } Z). \end{cases}$$



$\square$

$$\begin{array}{ccc} X & & x \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & & [s_0(x) : s_\infty(x)]. \end{array}$$

$B = s_0^{-1}(0) \cap s_\infty^{-1}(0)$

If blow up X along B

$\Rightarrow$ $\begin{array}{c} Bl_B X \\ \downarrow \\ \mathbb{P}^1 \end{array}$ is well-defined

Moreover, is locally modeled on $\mathbb{C} \times \mathbb{P}^1 \times B \to \mathbb{P}^1$.

Setup. (Symplectic Lefschetz pencil).

$(X, \omega) \Rightarrow$ closed symplectic mfld, $\quad \omega^n \neq 0$, $d\omega = 0$.

$B \subseteq X$ is smooth real codimension 4 symplectic submfld of X.

$\begin{array}{c} X \setminus B \\ \downarrow \\ S^2 \end{array}$ is smooth fibration away from $\{x_1, \cdots, x_k\} \subseteq X \setminus B$

image given by $\{y_1, \cdots, y_m\} \subseteq S^2$.

① $X \setminus B \big|_{S^2 \setminus \{y_1, \cdots, y_k\}}$ is a symplectic fibration

$$\begin{array}{c} X \setminus B \\ \pi \downarrow \\ S^2 \setminus \{y_1, \cdots, y_k\} \end{array}$$

$\left\{ \begin{array}{l} \text{fibers are symplectic submanifolds} \\ (\ker \pi_*)^{\perp}_{\omega} \xrightarrow{\pi_*} (TS^2|_{S^2 \setminus \{y_1, \cdots, y_k\}}, \omega_{std}) \end{array} \right)$

is an isomorphism.

② near $\{x_1, \cdots, x_k\} \subseteq X$, $\exists\ J$, compatible w/. ω, integrable,
(almost complex structure)

s.t. $\begin{array}{c} X \setminus B \\ \downarrow \\ S^2 \end{array}$ is locally modeled on $(z_1, \cdots, z_n) \mapsto z_1^2 + \cdots + z_n^2$.

Exercise. For (X, ω), $L \to X$ hol. line bundle,

then show that an (algebraic) Lefschetz pencil

can define a symp Lef. pencil.

Thm (Donaldson) For (X, ω) closed,

If $[\omega] \in \mathrm{Im}(H^2(X; \mathbb{Z}) \hookrightarrow H^2(X; \mathbb{R}))$.

Then $\exists$ symp Lef-pencil on X,

s.t. the Poincaré dual of a smooth fiber is $k[\omega]$
for some $k \gg 1$.

Our focus "Exact setting"

" Polynomial maps from a smooth affine algebraic variety over $\mathbb{C}$ ".

2. Vanishing cycles, Lefschetz thimbles, & monodromy

2.1 Local model

$$\begin{array}{l} \mathbb{C}^n = \mathbb{R}^{2n}, \quad \omega_{std} = \frac{i}{2}\sum_j dz_j\wedge d\bar{z}_j = \sum_{i=1}^{n} dx_i\wedge dy_i . \\ \big\downarrow \; z_1^2+\cdots+z_n^2 = \pi \\ \mathbb{C} \end{array}$$

Lemma (Exercise). $(T^*S^{n-1}, \sum dp\wedge dq)$ is symplectomorphic to $(\pi^{-1}(1), \omega_{std}|_{\pi^{-1}(1)})$. $\quad z=(z_1,\dots,z_n)$

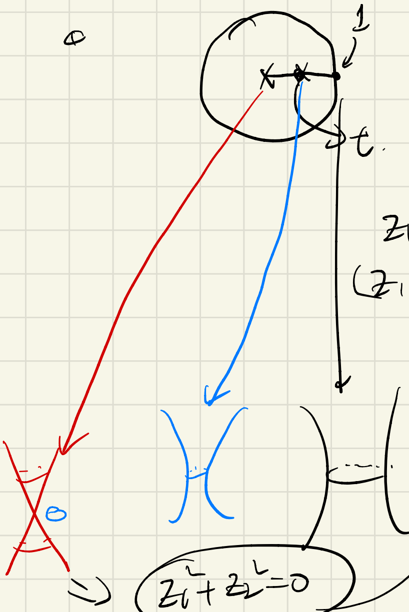
cot. coordinate

Hint. Consider the map $\pi^{-1}(1)\to T^*S^{n-1}$

$$(z_1,\dots,z_n)\mapsto \left(\frac{Re(z)}{\|Re(z)\|}, \|Re(z)\|\, Im(z)\right) \quad (*)$$

$$T^*S^{n-1}\hookrightarrow T^*\mathbb{R}^n \cong \mathbb{R}^n\times\mathbb{R}^n . \quad \square$$

Observation.



$\exists$ symplectomorphism $\pi^{-1}(t)\to T^*S^{n-1}_t$ in the spirit of $(*)$.

$n=2$

$$z_1^2+z_2^2=1 \cong \mathbb{C}^* \cong T^*S^1$$

$$\underset{s}{(z_1+\sqrt{-1}z_2)}\,\underset{t}{(z_1-\sqrt{-1}z_2)}=0 .$$

Def^n (local model)

$$S^{n-1}\overset{0}{\hookrightarrow} T^*S^{n-1}\cong \pi^{-1}(1)$$

is called the vanishing cycle.

In particular, S^{n-1} is Lagrangian!

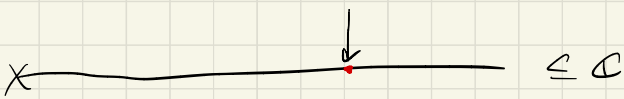
$$\begin{array}{c} \mathbb{C}^n \\ \downarrow \ \pi = z_1^2 + \cdots + z_n^2 \\ \mathbb{C} \end{array}$$

$$z_i = x_i + \sqrt{-1}\, y_i$$

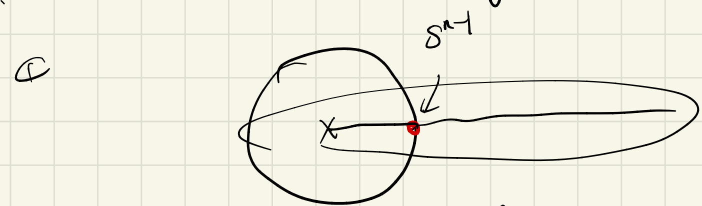
$$(x_1^2 - y_1^2) + \cdots + (x_n^2 - y_n^2)$$

<u>observe</u>. $Re(\pi)$ is a Morse function

<u>Def^n</u>. (Lefschetz thimble) The stable submfld of $Re(\pi)$.
L_0.



<u>(Exercise)</u> $L_0 \cap \pi^{-1}(1) = S^{n-1}$ (the vanishing cycle).



<u>(Exercise)</u> Show that L_0 is diff. $\mathbb{R}^n$; is Lagrangian.

<u>2.2. Exact Lefschetz fibration</u>.

<u>Def^n</u>. (X, ω, θ) $\quad$ X is a compact mfld with ∂

$\omega \in \Omega^2(X)$, symplectic

$\theta \in \Omega^1(X)$, $d\theta = \omega$.

<u>Liouville vector field</u>: Z $\quad$ $i_Z \omega = \theta$.

<u>Require</u>: Z is outward pointing along ∂Z. $\Rightarrow$ Liouville domain.

<u>Def^n</u>. (X, ω, θ) $d\theta = \omega$. $\quad$ $X \xrightarrow{\pi \text{ (smooth)}} \mathbb{C}$ $\quad$ π is called a Lefschetz fibration if
<u>(Lefschetz fibration, exact)</u>

(1) π has only finitely many critical points. is locally modeled on $\begin{array}{c}(z_1, \cdots, z_n) \\ \downarrow \\ z_1^2 + \cdots + z_n^2\end{array}$.

(2) For smooth fibers, $X_z := \pi^{-1}(z)$, $(X_z, \omega|_{X_z}, \theta|_{X_z})$ is a Liouville domain.

(3) $(\ker \pi_*)^{\perp}_{\omega} \xrightarrow{\pi_*} (T\mathbb{C}, \omega_{std})$ is an isomorphism.

(4) $\exists$ symplectic open embedding $\begin{pmatrix} \partial X_Z \times (1-\varepsilon, 1] \times \mathbb{C} \\ \theta_Z = \theta|_{\partial X_Z} \\ d(t\,\theta_Z) \oplus \omega^{std}_{\mathbb{C}} \end{pmatrix} \hookrightarrow X$

near a neighborhood of ∂X

s.t. the projection $X \xrightarrow{\pi} \mathbb{C}$ is given by $\partial X_Z \times (1-\varepsilon, 1] \times \mathbb{C} \to \mathbb{C}$

<u>Exercise</u>. Show that the parallel transport is well-defined over $\mathbb{C} \setminus \{\text{critical values of } \pi\}$.

$TX = T^{vir}X \oplus (T^{vir}X)^{\perp}$, $\quad (T^{vir}X)^{\perp} \xrightarrow{\pi_*} T\mathbb{C}$

$T^{vir}X$: tangent space of the fiber

$\Rightarrow$ can lift vector fields on $\mathbb{C}$ to $\cdots$ on X.

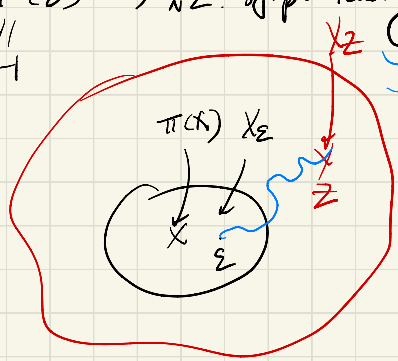
Suppose $x \in \mathrm{Crit}(\pi)$, locally modeled on $\mathbb{C}^n \xrightarrow{z_1^2 + \cdots + z_n^2} \mathbb{C}$.

$\varepsilon \neq 0$

$\pi^{-1}(\varepsilon) \hookrightarrow X_\varepsilon$ symplectically.

$\cong T^*S^{n-1}$

$S^{n-1} \hookrightarrow 0$



<u>Def^n</u>. Defines the vanishing cycle associated with x in the fiber X_ε.

"local play"

For another smooth fiber X_z, define the vanishing cycle by parallel transport.

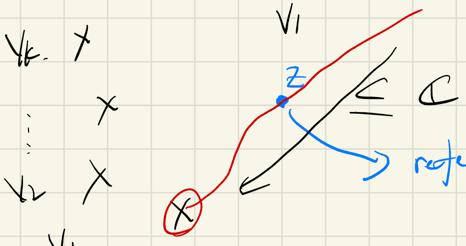
chapter 3

D. Seidel Picard-Lefschetz / Fukaya

<u>Exercise</u>. Different paths define the same Lag sphere up to Hamiltonian isotopy.

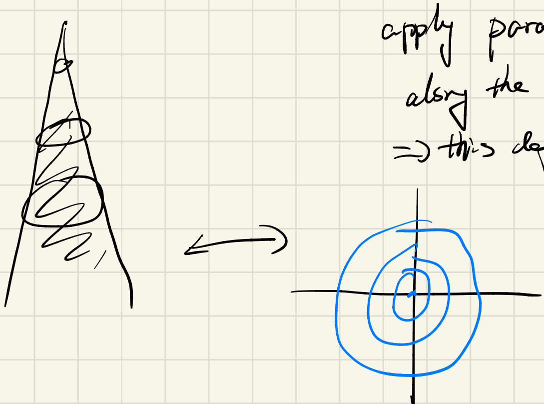
Def^n. Lefschetz thimble.

$X \xrightarrow{\pi} \mathbb{C}$ Critical values are distinct.



reference smooth fiber X_z

with vanishing cycles $V_1,\dots,V_k$



apply parallel transport of V_1 along the red line.

$\Rightarrow$ this defines thimble ("take the trace").

$\cong \mathbb{R}^n$

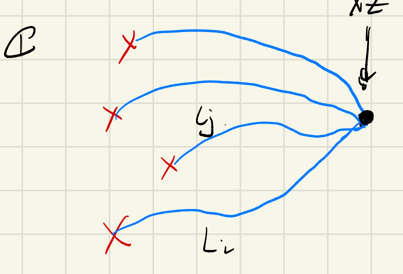
they are Lagrangians inside X.

Sum up. $X \xrightarrow{\pi} \mathbb{C}$ $F = X_z$ contains Lagrangian spheres $V_1,\dots,V_k$ (vanishing cycles).

X contains Lagrangian $\mathbb{R}^n$ $L_1,\dots,L_k$ (thimble)

$$L_i \cap F = V_i$$

2.3 Intersection pairing and Picard Lefschetz theory.



× ⇒ critical values.

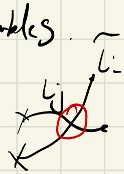
Def^n. (capped-off Lefschetz thimbles)

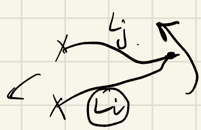
$\Leftrightarrow$ The trace of the parallel transport of the vanishing cycles along the blue line

$\Rightarrow$ Lagrangian discs inside X.

Def^n. L_i, L_j (capped-off) Lefschetz thimbles.

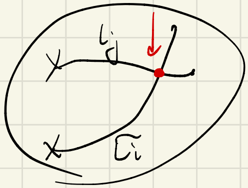
$$\#\underbrace{L_i \cap L_j}_{V_i \quad V_j} := \#\tilde{L}_i \cap L_j$$





Prop^n If $\tilde{L}_i \cap L_j \neq \phi$, then $L_i \cdot L_j = V_i \cdot V_j$.

proof.



$\tilde{L}_j \cap L_i = \phi$

key observation.

$\tilde{L}_i$ intersect L_j transversely
if and only if
the corresponding vanishing cycles
intersect transversely in the fiber.

The rest is to compute local orientation signs. $\square$

$L_i, L_j \subseteq X$

$V_i, V_j \subseteq F$